Waist estimates for the maps from the sphere

Alexey Balitskiy [1]

[1] balitski@mit.edu
Massachusetts Institute of Technology

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Fibers of maps dropping dim

$$f: \underbrace{S^n}_{\text{the round unit } n-\text{sphere}} \to \underbrace{Y^m}_{m-\text{dim. top. space}}, \quad m < n$$

Can one guarantee the existence of a fiber $F = f^{-1}(y), y \in Y^m$, large in some sense?

- in terms of volume (Gromov's waist)
- in terms of the diameter (Urysohn's width):

$$\mathrm{UW}_m(X) = \inf_{f:X \to Y^m} \sup_{y \in Y^m} \mathrm{diam}(f^{-1}(y))$$

- in terms of the Urysohn width

A fiber of large Urysohn width

Theorem (Gromov?)

1. *Let* $n = (m+1)(d+1)$. *Every map* $f : S^n \to Y^m$ *has a fiber* $F = f^{-1}(y)$ *of Urysohn d-width* $\mathrm{UW}_d(F) \geq \mathrm{UW}_{n-1}(S^n) > \frac{\pi}{2}$.
2. *Let* $n = (m+1)(d+1) - 1$, $\varepsilon > 0$ *(small). There is a map* $f : S^n \to \triangle^m$, *whose fibers all have* $\mathrm{UW}_d(F) < \varepsilon$.

Theorem

Assume that a generic PL-map $f : S^3 \to [0,1]$ *is such that all regular fibers* $f^{-1}(y), y \in (0,1)$, *are PL-isomorphic to* S^2. *Then there is a fiber* $F = f^{-1}(y)$ *of Urysohn* 1*-width* $\mathrm{UW}_1(F) \geq \frac{1}{2}$.

Conjecture: if $f : S^n \to Y^m$ is generic and almost all fibers have "bounded topological complexity", then $\exists F = f^{-1}(y)$ with $\mathrm{UW}_{n-m-1}(F) > \frac{1}{10^{10}}$.