WORKSHEET 8

Date: 10/20/2021

Name:

Induction and its Sisters

REMARK 1 (The First Principle of mathematical induction)**.** Suppose that for each positive integer n we have a statement $P(n)$. If we prove the following two things

1. $P(1)$ is true;
2. for all n, if $P(n)$ is true then $P(n+1)$ is also true;
   then $P(n)$ is true for all positive integers n.

REMARK 2 (The Second Principle of mathematical induction)**.** Let k be an integer. Suppose that for each integer $n \geq k$ we have a statement $P(n)$. If we prove the following two things

1. $P(k)$ is true;
2. for all $n \geq k$, if $P(n)$ is true then $P(n+1)$ is also true;
   then $P(n)$ is true for all positive integers $n \geq k$.

REMARK 3 (The Strong Form of Induction)**.** Suppose that for each integer $n \geq k$ we have statement $P(n)$. If we prove the following two things:

1. $P(k)$ is true; and
2. for all $k \leq n$, if $P(k), P(k+1), \ldots P(n)$ are all true, then $P(n+1)$ is true;
   then $P(n)$ is true for all $n \geq k$.

What is the distinction of the first principle of mathematical induction, the second principle of mathematical induction and strong induction? (You should be able to answer this question in Wednesday's section.)

Getting Our Hand's Dirty

Please partition your self's into give groups. One of your group members must know LaTeX. You will submit your proof to me and I will post it online for your peers to view. You can name your group whatever you want, or choose from the following list: Chelsea FC, Dodger Nation, Raider Nation, Chivas, or Team U.S.A. If you choose a team name, it must be appropriate. I will give the approval in section. Write the answers to each question before you start the proof.

- What principle of induction are you going to use? Why did you choose this particular induction?

- What is your induction hypothesis?

- What are you trying to prove?

- Give a short paragraph of the strategy regarding your proof.

1. Prove that for every nonnegative integer n, $(4^{3n} - 1)$ is a multiple of 9.
   Hint: Use induction.

   Proof:

2. Example: If $\{x_n\}$ is a sequence defined recursively by $x_1 = 1, x_2 = 3$, and $x_n = 2x_{n-1} - x_{n-2}$ for $n \geq 3$, then $x_n = 2n - 1$ for all natural numbers n.

   Proof:

3. Define a sequence $\{x_n\}$ recursively by $x_1 = 1, x_2 = 4$, and $x_n = 2x_{n-1} - x_{n-2} + 2$ for $n \geq 3$. Conjecture a formula for x_n and verify that your conjecture is correct. i.e prove the formula holds by induction.

   Proof:

4. First, we define $\sum =$ 🎃. Prove that

$$\mathop{\text{🎃}}_{k=1}^{n} \frac{1}{k^2} < 2.$$

Hint: Prove the stronger statement that the left hand side is less than $2 - \frac{1}{n}$.

Proof:

5. Assume n is a non negative integer. Prove $3|(n^3 - n)$.

   (Hint: Just because induction is a choice doesn't always mean it is the easiest choice. Note that $n^3 - n$ factors to $(n-1)n(n+1)$. So, if you show that the product of three consecutive integers is a multiple of 3 then the statement above is proven. If you approach the problem this way you will need the following lemma: If $3|a$ and $k \in \mathbb{Z}$ then $3|ak$.)

   Proof: