Worksheet 13

Date: 8/11/2021

Name:

Partitions and Functions

DEFINITION 1. Let n be a positive integer, and let S be a set. A **partition** of S is a collection of subsets $S_1, S_2, \ldots, S_k$ such that each element of S lies in exactly one of these subsets. In other words,

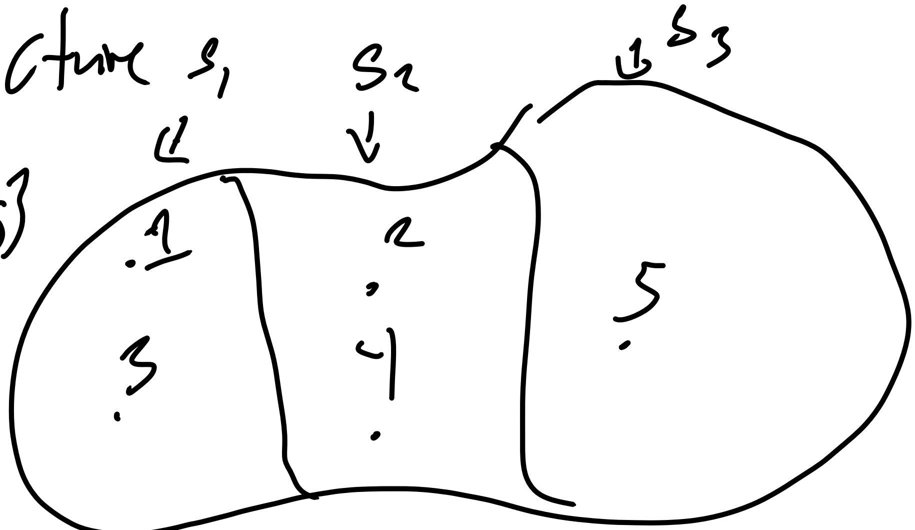
$$S = S_1 \cup S_2 \cup \ldots \cup S_k \text{ and } S_i \cap S_j = \phi \text{for any } i \neq j.$$

THEOREM 1. *Let S be a set and $\sim$ be an equivalence relation on S. Then, the equivalence classes of $\sim$ form a partition of S.*

REMARK 2. Note that the converse is also true. Let $P = \{A_\alpha : \alpha \in I\}$ be a partition of a nonempty set A. Then there exists an equivalence relation R defined on A such that $P = \{[a] : a \in A\}$ is the set of equivalence classes determined by R.

Partition picture

$S = \{1, 2, 3, 4, 5\}$



$S_1 \cup S_2 \cup S_3 = S$ and $S_i \cap S_j = \phi$ for $i \neq j$

1. A relation R is defined on $\mathbb{Z}$ by xRy if $11x - 5y$ is even. Then R is an equivalence relation.

   (a) Show R is an equivalence relation. Solution: You can show this.

   (b) Determine the equivalence class of the relation R.

First $0 \in \mathbb{Z}$, so lets find $[0]$. $[0] = \{x \in \mathbb{Z} \mid x R 0\}$ $= \{x \in \mathbb{Z} : 11x$ is even$\} = \{x \in \mathbb{Z} : x$ is even$\}$ $= \{0, \pm 2, \pm 4, \ldots\}$. Recall that the distinct e.c. always produce a partition. since 1 is odd $1 \notin [0]$. $[1] = \{x \in \mathbb{Z} \mid xR1\} = \{x \in \mathbb{Z} \mid 11x - 5$ even$\}$ $\Rightarrow x$ is odd $[x] = \{\pm 1, \pm 3, \ldots\}$. $\boxed{[0], [1]}$.

2. Let $A = \mathbb{Z}$. Find four different partitions of the integers.

$A_1 = \{$the set of all even integers$\}$
$B_1 = \{$the set of all odd integers$\}$
$A_1 \cup B_1 = \mathbb{Z}$

$A_1 = \{3k \mid k \in \mathbb{Z}\}$, $A_2 = \{3l+1 \mid l \in \mathbb{Z}\}$, $A_3 = \{3j+2 \mid j \in \mathbb{Z}\}$

$A_1 \cup A_2 \cup A_3 = \mathbb{Z}$

$A_0 = \{0\}$, $A_1 = \{\pm 1, \pm 2, \ldots \pm n \ldots\}$ wh $n \neq 0\}$

$A_0 \cup A_1 = \mathbb{Z}$ $\quad A_0 \cap A_1 = \emptyset$ ✓ and $A_0 \neq \emptyset$ and $A_1 \neq \emptyset$.

$A_1 = \{1, 2, 3, \ldots\} = \{n \in \mathbb{Z} \mid n \neq 0\}$.
$A_2 = \{-1, -2, -3, \ldots\} = \{-|n| \in \mathbb{Z} \mid n \neq 0 \; n \in \mathbb{Z}\}$.

$A_3 = \{0\}$

$f : X \to Y$

DEFINITION 2. Let X and Y be sets. A **function** f from X to Y is a *rule* that assigns to each element $x \in X$ a single element of Y, denoted by $f(x)$ or $y = f(x)$.

DEFINITION 3. If $f : X \to Y$ is a function, then

- The set X is called the **domain** of f;
- The **image** of f is the set of all elements of Y that are equal to $f(x)$ for some $x \in X$. We write $f(X)$ for the image of f, and it is specifically,

$$f(X) = \{f(x) | x \in X\}.$$

Let X be any set. We define a function $i_x : X \to X$, called the *identity* of X by

$$i_x(x) = x$$

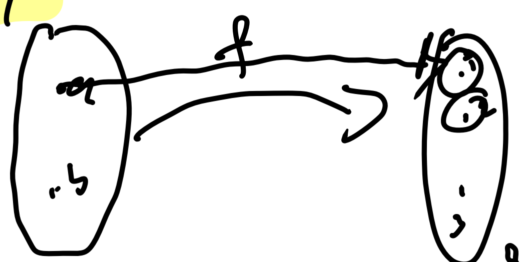
DEFINITION 4. Let $f : X \to Y$ be a function. We say that

- f is **onto** or **surjective** if the image $f(X) = Y$. i.e, for every $y \in Y$, there exists an $x \in X$ such that $f(x) = y$.
- f is **one-to-one** or **injective** if for $a, b \in X$

  if $f(a) = f(b)$ then $a = b$. $(\Leftrightarrow)$ $a \neq b \Rightarrow f(a) \neq f(b)$

- f is a **bijection** if f is both surjective and injective.

Ex: 1

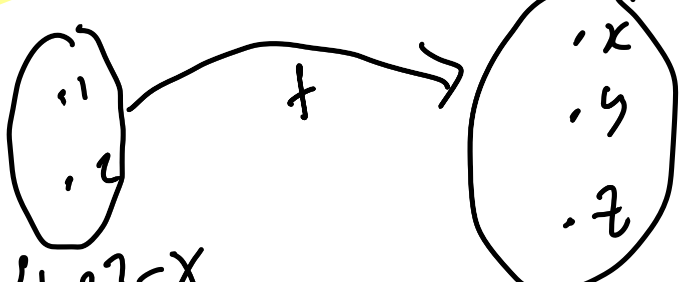


$X = \{a, b\}$ $Y = \{1, 2, 3\}$

$f(a) = \{1, 2\}$

Not a function

Ex: 2



$\{1, 2\} = X$ $\{x, y, z\} = Y$

$f(1) = x$
$f(2) = x$

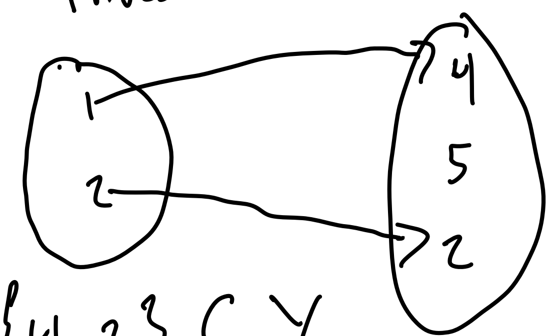
$f(X) = \{x\}$

Ex: 3 $X = \{1, 2\}$ $\quad Y = \{4, 5, 2\}$

$f(1) = 4$

$f(2) = 2$

so f is injective but not surjective

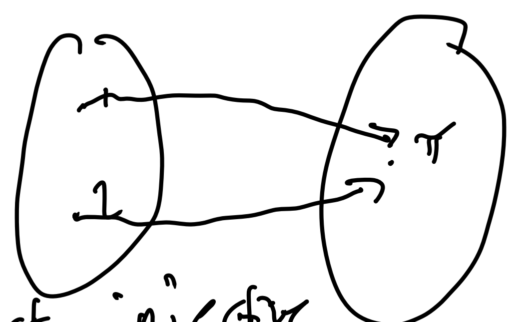


$f(X) = \{4, 2\} \subset Y$

Ex: 4 $X = \{1, 2\}$ $\quad Y = \{\pi\}$

$f(1) = \pi$

$f(2) = \pi$



f is surjective but not injective.

1. Prove $f:\mathbb{R}\to\mathbb{R}$ is a bijection when

$$f(x)=\frac{3x+5}{7}$$

pf: To show f is a bijection, we show f is first injection then we show f is surjection.

1-1: suppose $x, y \in \mathbb{R} \ni f(x) = f(y)$.

Then $$\frac{3x+5}{7} = \frac{3y+5}{7}$$

$$\iff 3x+5 = 3y+5$$

$$\iff 3x = 3y$$

$$\iff x = y.$$

So, by definition f is injective.

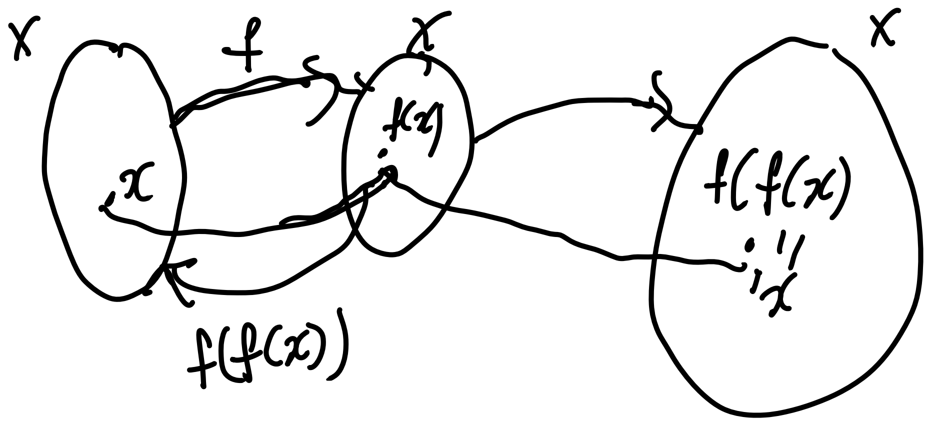
Scratch work

surjective: Let $y \in \mathbb{R}$, goal find $x \in \mathbb{R} \ni f(x) = y$. $\frac{3x+5}{7} = y \iff 3x+5 = 7y \iff x = \frac{7y-5}{3} \in \mathbb{R}$

Let $x = \frac{7y-5}{3}$. Since $x \in \mathbb{R}$ $f(x) = f\left(\frac{7y-5}{3}\right)$

$$= \frac{3\left(\frac{7y-5}{3}\right)+5}{7} = y.$$ So, f is surjective.

2. Let X be a set and $f : X \to X$ be a function such that $f(f(x)) = x$ for all x. Then f is a bijection.



Pf: suppose $x, y \in X$ such that $f(x) = f(y)$

goal: prove $x = y$.

now, $f(x), f(y) \in X$. so, since f is a function $f(f(x)) = f(f(y)) \iff x = y$ by def of f.

Hence, f is 1-1 or injective

now, let $x \in X$: goal: find $z \in X \ni f(z) = x$.

Then $f(x) \in X$, and $f(f(x)) = x$ by def of f.

so let $f(x) = z$. Then $f(z) = f(f(x)) = x$

so f is surjective. Hence f is bijective.