worksheet $\underline{\overline{XV}}$ (15)

Recall from lecture

Def : Let $f: X \to Y$ be a function

we say that

a) f is surjective (onto) if for every $y \in Y$, there exists $x \in X$ such that $f(x) = y$

b) f is injective (one-to-one) if $\forall x, y \in X$ if $f(x) = f(y)$ then $x = y$.
$\iff$ if $x \neq y$ then $f(x) \neq f(y)$.

c) f is bijective iff f is injective and surjective.

Ex: Let $X = \{1, 2, 3\}$ and $Y = \{a, b\}$

a) How many <u>functions</u> $f: X \to Y$ are there?

$|X| = 3$

$|Y| = 2$

$$f: \quad \begin{array}{l} 1 \to a \\ 1 \to b \\ 2 \to a \\ 2 \to b \\ 3 \to a \\ 3 \to b \end{array}$$

$\underline{\ 2\ }$ $\underline{\ 2\ }$ $\underline{\ 2\ }$

Total $2^3 = 8$ total functions

If $|X| = n$, $|Y| = m$ $f: X \to Y$.

How many functions exists?

m^n

b) How many of these functions $f: X \to Y$ are injective?

$X = \{1, 2, 3\}$

$Y = \{a, b\}$

injective $f(x) = f(y) \Rightarrow x = y$.

If we want f to be injective $f: X \to Y$

$|Y| \geq |X|$

Note:

$f: X \to Y$

$1 \to a$

$2 \to b$

$3 \to a$ or b

so none!

c) How many functions $f: X \to Y$ are surjective?

surjective: $\forall y \in Y, \exists x \in X \ni f(x) = y$.

$f: X \to Y$

$1 \to a$ or b

$2 \to$

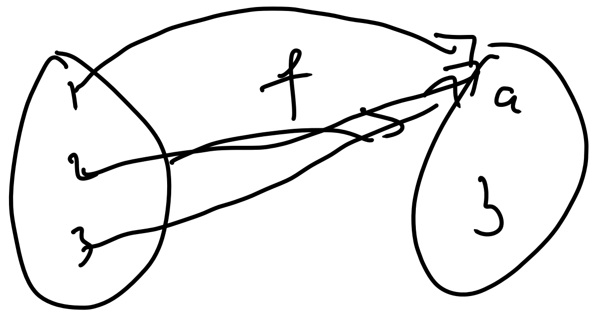
$3 \to$

$\underset{\text{choice to send 1}}{\underline{2}} \ \underset{\text{choices 2 to the other}}{\underline{1}} \ \underline{2} + \underset{\text{2 choices to send 1}}{\underline{2}} \ \underset{\text{send 2 to the other}}{\underline{1}} \ \underset{\text{send 3 to other}}{\underline{1}}$

$4 + 2 = 6$

Better method: #(Total functions) − #(non-surjective functions) = #(surjective functions)

$8 - 2 = 6$



$f(X) = \{a\}$

$f(X) = \{b\}$

Now, let $g: Y \to X$ when $X = \{1, 2, 3\}$, $Y = \{a, b\}$

a) How many functions exist $g: Y \to X$?

$\underline{3}\ \underline{3} = 9$, $|X|^{|Y|} = 3^2$

b) How many of these are injective?

$\underline{3}\ \underline{2} \to$ $\boxed{6 \text{ choices}}$

3 choice where to send a

2 choices where to send b

c) How many are surjective $\qquad |X| = 3$, $|Y| = 2$

$g : Y \longrightarrow X$

$a \to 1$

$b \to 2$

$0 \qquad 3$ nothing is sent to 3.

So no surjective function.

Let $h : A \to B$ be a function and $|A| < |B|$ then f cannot be surjective.

Composition of functions

Let X, Y, Z be sets & $f : X \to Y$ & $g : Y \to Z$ then the composition of f & g is the function $g \circ f : X \to Z$

$\ni \; g \circ f(x) = g(f(x))$

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend right, "g \circ f"'] & Y \arrow[r, "g"] & Z \end{tikzcd}$$

$x \xrightarrow{f} f(x) \xrightarrow{g} g(f(x))$

Ex: if $f: X \to Y$ and $g: Y \to Z$ are functions prove (1) If f, g injective, then $g \circ f$ injective.

Pf: let $x_1, x_2 \in X$ and assume

$g \circ f(x_1) = g \circ f(x_2)$.

$\Rightarrow g(f(x_1)) = g(f(x_2))$ [since g is injective]

$\Rightarrow f(x_1) = f(x_2)$ [since f is injective]

$\Rightarrow x_1 = x_2$ $\square$

(2) If f, g surjective, then $g \circ f$ surjective.

goal: let $z \in Z$. we want to show there exist an $x \in X \ni z = g \circ f(x)$.

Pf: let $z \in Z$. since g is surjective $\Rightarrow$ $\exists y \in Y \ni g(y) = z$. since f is surjective $\exists x \in X \ni f(x) = y$. now

$z = g(y) = g(f(x)) = g \circ f(x)$. $\square$

3) prove that if $g \circ f$ is injective then f is injective.

pf Let $x, y \in X \ni f(x) = f(y)$.

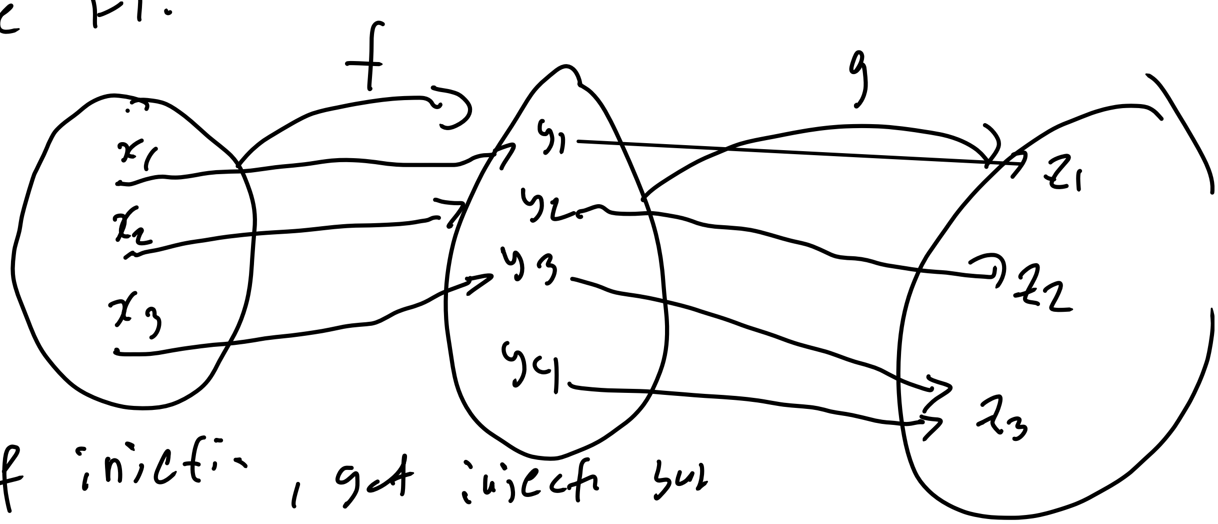
so $g(f(x)) = g(f(y))$ since g is a function

$\Rightarrow g \circ f(x) = g \circ f(y)$

$\Rightarrow x = y$ since $g \circ f$ is injective.

so f is injective. $\square$

Ex: $g \circ f$ 1-1 $\Rightarrow f$ 1-1 but g need not be 1-1.



f injective, $g \circ f$ injective but g not injective.

ex: You can prove

If $g \circ f$ onto, then g is onto.

$f: X \to Y$, $g: Y \to Z$ f, g functions