WORKSHEET 5

Date: 10/11/2022

Name:

PROOFS AND NEGATING STATEMENTS

As the title suggests, we will go over proofs and negation of statements in section today. I found a web-page which goes over basic proof techniques. I suggest you click the following link and explore the page. I would love to go over this in section, but unfortunately we don't have enough time. Click me please.

NEGATING STATEMENTS

- We write $\exists$ to mean "there exists". Example: $\exists x \in \mathbb{Z} \ni x+1=0$ reads "there exists x in the integers such that $x+1=0$.
- We write $\forall$ to mean "for all". Example: $\forall x \in \mathbb{Z}$, $-x \in \mathbb{Z}$ reads " for all x in the integers, $-x$ is in the integers".
- What is the negation of the statements above?
- Negation of $\leq$ is $>$

1. Negate the following statements:

   (a) $\forall n \in \mathbb{Z} \ni$ if n is prime, then n is odd.
   Solution: $\exists n \in \mathbb{Z} \ni$, n is prime and n is even.

   (b) $\exists x, y \in \mathbb{Z}$ such that $x+y \notin \mathbb{Z}$.
   Solution: $\forall x, y \in \mathbb{Z}$ such that $x+y \in \mathbb{Z}$.

   (c) $\exists x \in \mathbb{Z}$ such that $\forall n \in \mathbb{Z},\ x \neq n^2+2$
   Solution: $\forall x \in \mathbb{Z}$, $\exists n \in \mathbb{Z}$, such that $x = n^2+2$

   (d) Let A and B be non empty sets. What is $\sim (A \subseteq B)$. Recall the definition of a subset:

   $$\forall x \in A, x \in A \Rightarrow x \in B.$$

   Solution: $\exists x \in A$ and $x \notin B$.

Perfect Proof Practice

Break into groups and construct a proof for the following questions. You group will volunteer one represented, Squid Game style, to attempt a proof on the chalkboard.

1. Let x be a real number. Prove that if x is irrational and $x \geq 0$, then $\sqrt{x}$ is irrational.

   Proof. The proof is by contradiction. Let x be a real number and assume $\sqrt{x}$ is a rational number. Hence, there exists an integer a and a non zero integer b such that $\sqrt{x} = \frac{a}{b}$. Squaring both sides gives $x = \frac{a^2}{b^2}$. But a^2 is an integer and b^2 is also an integer, in particular it is a non negative integer. Hence, x must be rational. This contradicts our assumption that x is irrational, so it must be the case that $\sqrt{x}$ is irrational. This proves the statement.

   $\square$

2. Prove that $\{x \in \mathbb{Z} : 18|x\} \subseteq \{x \in \mathbb{Z} : 6|x\}$.
   [Notation: we say $18|x$ if and only if there exists an integer z such that $x = 18z$. In general if $a \neq 0, b$ are integers then we say $a|b$ if and only if there exists an integer c such that $b = ac$.]

   Proof. Let $y \in \{x \in \mathbb{Z} : 18|x\}$. Then there exists an integer k such that $y = 18k$. Doing basic algebra gives $y = 6(3k)$ and since $3k$ is an integer $6|y$ by definition. Hence, $y \in \{x \in \mathbb{Z} : 6|x\}$. Since, y was arbitrary this proves the statement.

   Note: this is a proper subset, since $6 \in \{x \in \mathbb{Z} : 6|x\}$ but $6 \notin \{x \in \mathbb{Z} : 18|x\}$. Assume we have equality of sets. Then there exists a $l \in \mathbb{Z}$ such that $6 = 18l$. Then $1 = 3l$. But this means that l must be equal to $1/3$ which is not an integer. This contradicts the choose of l being an integer. Hence this shows our set must be a proper subset.

   $\square$

3. Let $a \in \mathbb{Z}$. Prove that if $7a+8$ is odd if and only if a is odd.

   Proof. We prove the converse first. Assume a is odd. Then there exists an integer k such that $a = 2k+1$. Then $7a+8 = 7(2k+1)+8 = 2(7k+7)+1$. Since $7k+7$ is an integer, this shows $7a+8$ is odd.

   We know show the forward direction, but we do this by showing the contrapositive. Assume a is an even integer. Then there exists an integer m such that $a = 2m$. Then $7a+8 = 7(2m)+8 = 2(7m+4)$. Since $7m+4$ is an integer, this shows $7a+8$ is even. This now shows the result.

   □

4. Using a proof by contradiction show **there does not exist a smallest positive rational number**.

   Proof. The proof is by contradiction. Assume there exists a smallest positive rational number, i.e $\exists p \in \mathbb{Q}^+$ such that $0 < p \leq q$ for all $q \in \mathbb{Q}^+$. Consider the rational number $\frac{p}{2}$. Clearly this is a rational number which is greater than zero and $\frac{p}{2} < p$. So $0 < \frac{p}{2} < q$ for all $q \in \mathbb{Q}^+$. This contradicts p being the smallest non negative rational number with this property.

   □

5. Suppose $x \in \mathbb{R}$. Prove that if $x^2+5x < 0$, then $x < 0$.

   Proof. We will show the contrapositive of the result. Let x be a real number and $x \geq 0$. Then $5x \geq 0$ and we know $x^2 \geq 0$ for any real number x. Adding these two inequalities we get $x^2+5x \geq 0$. This shows the result. □

Here is another proof of this result. We use a direct proof.

Proof. Assume $x^2+5x < 0$. Then $x(x+5) < 0$. For this to be the case, we have to cases to consider. The first case, we must have $x < 0$ and $x+5 > 0$. In the second case we must have $x > 0$ and $x+5 < 0$.
Case 1. Assume $x < 0$ and $x+5 > 0$. This shows the result.
Case 2. Assume $x > 0$ and $x+5 < 0$. Then $x < -5 < 0$. But this is a contradiction, since no number is greater than zero and less than zero at the same time. Hence, it must be the case that $x < 0$. □