WORKSHEET 2

Date: 10/03/2024

Name:

Implication Statements and Negations

Math 8 - Section 1.2 - 1.3 Worksheet

Key Terms: Tautology, Contradiction, Conditional sentence, antecedent, consequent,Converse, Contrapositive, Biconditional sentence, Open sentence/Predicate,Truth set, Universe of discourse,equivalent,Existential quantifier, Universal quantifier, Unique existential quantifier.

Lets look at the following question: What numbers satisfy the following equation?

$$z^3 + 1 = 0$$

As you would have guessed this is a vague question and is not a great "mathematical question". In other words the statement $p(z) :=$ "$z^3 + 1 = 0$" is not a propositional statement. Is there a way to make this into a propositional statement?

So how do we remedy the remark above? Well first, we should pick our solutions of this equation to come from a particular source. In the language of the book we call it a "Universe of discourse". But we still have an issue. An idea what it is? Hint: we want a proposition to be either true or false.

Lets see if we can answer the previous question with some examples. Let $p(z)$ be the open sentence above with our universe of discourse equal to the natural numbers, $\mathbb{N} = \{1, 2, 3, 4 \ldots\}$.

1. There exists a natural number that satisfies $p(z)$.
2. For every natural number $p(z)$ is true.

Let $p(z)$ be the open sentence above with our universe of discourse equal to the natural numbers,
$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, 3, 4 \ldots\}$.

1. There exists an integer z such that $p(z)$ is true.
2. For every integer z, $p(z)$ true.
3. There exist two integers which make $p(z)$ true.

What do all these questions have in common? How can we make an open sentence into a proposition (in most cases)?

Negation

- We write $\exists$ to mean "there exists". Example: $\exists x \in \mathbb{Z} \ni x+1=0$ reads "there exists x in the integers such that $x+1=0$.
- We write $\forall$ to mean "for all". Example: $\forall x \in \mathbb{Z}, -x \in \mathbb{Z}$ reads " for all x in the integers, $-x$ is in the integers".
- Whats the negation of the statements above?
- Negation of $\leq$ is $>$

1. Negate the following statements:

   (a) $\exists x, y \in \mathbb{Z}$ such that $x+y \notin \mathbb{Z}$.

   (b) $\exists x \in \mathbb{Z}$ such that $\forall n \in \mathbb{Z},\ x \neq n^2+2$

   (c) Let A and B be non empty sets. What is $\sim (A \subseteq B)$. Recall the definition of a subset:

   $$\forall x \in A, x \in A \Rightarrow x \in B.$$

Practice Problems

1. What can be said about the truth value of Q when

   (a) P is false and $P \Rightarrow Q$ is true?

   (b) P is false and $P \Leftrightarrow Q$ is true?

2. Give, if possible, an example of a true conditional sentence for which

   (a) the converse is false.

   (b) the contrapositive is true.

3. Which of the following conditional sentences are true? Hint: put the statements in conditional form i.e. $P \Rightarrow Q$. Also, label what the statements P and Q are.

   (a) If Euclid's birthday was April 2, then rectangles have four sides.

   (b) 5 is prime if $\sqrt{2}$ is not rational.

   (c) 1+1=2 is sufficient for $3 > 6$.

4. Complete the truth table for the propositional form:

P	Q	R	$P \wedge Q$	$Q \wedge R$	$(P \wedge Q) \vee (Q \wedge R)$	$P \vee R$	$(P \wedge Q) \vee (Q \wedge R) \Rightarrow (P \vee R)$
T	T	T					
T	T	F					
T	F	T					
T	F	F					
F	T	T					
F	T	F					
F	F	T					
F	F	F					

5. Determine whether each of the following is a tautology, a contradiction, or neither.

   (a) $P \Rightarrow Q \Leftrightarrow P \wedge \sim Q$.

   (b) $P \wedge (Q \vee \sim Q) \Leftrightarrow P$.

6. Let $x \in \mathbb{Z}$. If x is even, then x^2 is even. Give the contrapositive, inverse and converse of the statement. State which are true and which are false.