WORKSHEET 0

Date: 09/26/2024

Name:

Introduction

Tennis, football, baseball and hockey may all be exciting games but to play them you must first learn (some of) the rules of the game. Mathematics is no different.

This class is an axiomatic subject. We start with a set of axioms and we use these axioms to prove propositions and theorems. It is extremely important to develop your skill at writing proofs.

Why are proofs so important? Suppose our task were to construct a building. We would start with the foundations. In our case these are the axioms or definitions – everything else is built upon them. Each theorem or proposition represents a new level of knowledge and must be firmly anchored to the previous level. We attach the new level to the previous one using a proof. So the theorems and propositions are the new heights of knowledge we achieve, while the proofs are essential as they are the mortar which attaches them to the level below. Without proofs the structure would collapse.

So what is a mathematical proof?

A mathematical proof is a watertight argument which begins with information you are given, proceeds by logical argument, and ends with what you are asked to prove. You should begin a proof by writing down the information you are given and then state what you are asked to prove. If the information you are given or what you are required to prove contains technical terms, then you should write down the definitions of those technical terms. Every proof should consist of complete sentences. Each of these sentences should be a consequence of

(i) what has been stated previously or

(ii) a theorem, proposition or lemma that has already been proved. In this book you will see many proofs, but note that mathematics is not a spectator sport. It is a game for participants. The only way to learn to write proofs is to try to write them yourself

What is this class about?

1. Here are some standard questions that you might want to consider.

   (a) Are there infinitely many prime numbers?
   Answer: Yes, this statement is true. We will prove this later by contradiction. In fact, I believe all the proofs of this statement are via contradiction.

   (b) Is it true that for every odd integer n, $n^2 - 1$ is divisible by 8? Answer: This statement is true. We prove this via a direct proof.

   (c) If p is a prime, then for any integer a, $a^p - a$ is a multiple of p. Answer: This statement is true. We might not have time to prove it, but I will give you a sketch week 8.

   (d) Are there more integers or real numbers? Answer: there are more real numbers than integers. We will answer this question in chapter 5 of the text.

   (e) Can you write every **even** positive integer, greater than 2, as a sum of two primes? Answer: We don't know. This is called Goldbach's conjecture.

2. Are there three consecutive integers whose product is a perfect square?
   Let x be an integer. The question ask if you can find an integer y such that

$$y^2 = x(x+1)(x+2)$$

   (a) What about positive integers? This is a hard question to answer, but the answer is no.

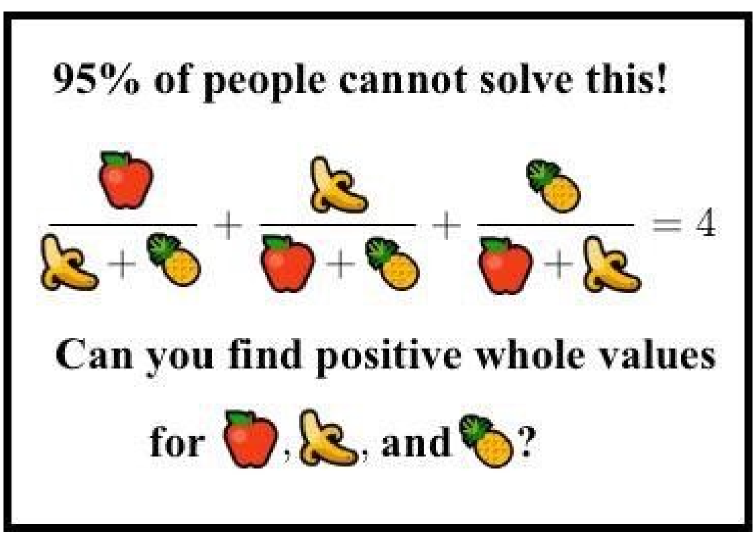
   (b) What if you consider non-zero rational solutions? This is an even harder question to answer. The answer is also no.

   (c) What if you consider real numbers?
   This is true. Let $x = 1$ then $y = \sqrt{6}$ is a solution.

3. Are there three integers that differ by 5, i.e, $x, x+5$ and $x+10$, and whose product is a perfect square?
   Let x be an integer. The question ask if you can find an integer y such that

$$y^2 = x(x+5)(x+15)$$

4. Look at this question.



5. What is a proof? We prove the following statement.

   Let n and m be integers. If n and m are even then $m+n$ is also an even integer.

   Proof. Assume m and n are even integers. Then, by the definition of an even integer, there exists an integer k and l such that $m = 2k$ and $n = 2l$. Now, $m+n = 2k+2l = 2(k+l)$. Since k and l are integers, their sum is guaranteed to be an integer. Let $c = k+l$. Then $m+n = 2c$. Thus, by the definition of an even integer, $m+n$ is even. □

   What is not a proof? This is not a proof of the previous statement.

   Let n and m be integers. If n and m are even then $m+n$ is also an even integer.

   Proof. because m and n are even integers we know the sum must also be even. by the definition of an even integer, $m+n$ is even. □

Summary of the lesson.
In section today we talked about how to best approached this class and what a proof is. We also spent some time going over some mathematical statements and trying to figure out if the statements were true or false. We weren't able to prove anything yet, but we hinted on how the proof should look and why some statement seemed to be true. To succeed in this class we must be familiar with definitions, memorize them, make them part of your arsenal when solving a problem. Math is hard, and we are going to struggle at times, but we must remember that learning takes time. We have to be comfortable with failing, but learn from our mistakes. If we improve 1% everyday we will becoming one step closer to achieving our goal.