Math 8 Week 1
June 25, 2021

Quiz 2

1. (3 points) Negate the following statement:

$$\forall \epsilon > 0, \exists \delta > 0, \text{ such that } \forall x \in \mathbb{R}, (|x| < \delta \Rightarrow x^2 < \epsilon).$$

$$\sim (\forall \epsilon > 0, \exists \delta > 0, \text{ such that } \forall x \in \mathbb{R}, (|x| < \delta \Rightarrow x^2 < \epsilon))$$

$$\exists \epsilon > 0 \text{ such that } \sim (\exists \delta > 0, \text{ such that } \forall x \in \mathbb{R}, (|x| < \delta \Rightarrow x^2 < \epsilon))$$

$$\exists \epsilon > 0 \text{ such that } \forall \delta > 0 \sim (\forall x \in \mathbb{R}, (|x| < \delta \Rightarrow x^2 < \epsilon))$$

$$\exists \epsilon > 0 \text{ such that } \forall \delta > 0, \exists x \in \mathbb{R}, \sim (|x| < \delta \Rightarrow x^2 < \epsilon)$$

$$\exists \epsilon > 0 \text{ such that } \forall \delta > 0, \exists x \in \mathbb{R} \text{ such that } |x| < \delta \text{ and } x^2 \geq \epsilon$$

2. (2 points) Prove or give a counterexample to the following: Let a, b be integers and a non zero. If $\frac{b^2}{a}$ is an integer then $\frac{b}{a}$ is an integer.

Proof. Here is a counterexample. Consider $a = 4$ and $b = 2$. In this case, $\frac{b^2}{a} = 1$ which is an integer but $\frac{b}{a} = \frac{1}{2}$ which is not an integer. □

3. (2 points) Prove no odd integer can be expressed as the sum of three cubes.

Proof. Assume, to the contrary, that there exists an odd integer n which can be expressed as the sum of three even integers x, y and z. Then $x = 2a$, $y = 2b$, and $z = 2c$ with $a, b, c \in \mathbb{Z}$. Therefore,

$$n = x + y + z = 2a + 2b + 2c = 2(a + b + c).$$

Since $a + b + c$ is an integer, n is even. This contradicts our assumption that n is an odd integer. □

4. Circle True or False for the following questions:

   (a) There exists integers a and b such that $(a + b)^2 = a^2 + b^2$.
   This statement is True. Assume $(a + b)^2 = a^2 + b^2$. Expanding the left side and simplifying we get $2ab = 0$. This forces either a to be zero or b to be zero. So consider $a = 1$ and $b = 0$.

   (b) For every integer x, there exists an integer y such that $y < x$.
   This statement is True.Let $x \in \mathbb{Z}$. Then the integer $y = x - 1$ has the desired property that $y < x$.