WORKSHEET 12

Date: 11/17/2021

Name:

Finishing 13.1(b)

1. Recall from last time that I attempted to use Fermat's Little theorem to show this result. It didn't follow immediately so I abandon the approach for the sake of time. Well here is the complete solution. We should that

$$r \equiv 2 \mod \{3,5\}, \text{and } r \equiv 27 \mod 43.$$

By a proposition from the previous week we get

$$r \equiv 2 \mod 15$$

. So know we are asking the following question: can we find r with $0 \leq r \leq 645$ such that

$$r \equiv 2 \mod 15$$

$$r \equiv 27 \mod 43$$

This is now similar to how we solved the first problem in our worksheet last week.

Cartesian product and Equivalence Relations

DEFINITION 1. Let A and B be sets. The **Cartesian Product** of A and B, written $A \times B$, is the set consisting of all the form (a,b) with $a \in A$ and $b \in B$.

Such a form (a,b) is called **ordered pair** of elements of A and B. Two ordered pairs are equal if each components are equal.

Example 1. Let $A = \{1,2\}$ and $B = \{0,1\}$. Find $A \times B$ and $B \times A$. From this example, does $A \times B = B \times A$?

DEFINITION 2. Let S be a set. A **relation** on S is defined as follows

1. Choose a subset R of the Cartesian product $S \times S$ i.e R consists of some of the ordered pairs (a,b) with $a,b \in S$.
2. For those ordered pairs $(a,b) \in R$, we write

$$a \sim b,$$

   and say a is **related** to b.

DEFINITION 3. Let S be a set and let $\sim$ be a relation on S. Then we say that $\sim$ is an **equivalence relation** if the following three properties hold for all $a,b,c \in S$:

1. Reflexive: $a \sim a$.
2. Symmetric: If $a \sim b$, then $b \sim a$.
3. Transitive: If $a \sim b$ and $b \sim c$, then $a \sim c$.

1. A relation R is defined on $\mathbb{Z}$ by xRy, this means $x \sim y$, if $x + 3y$ is even. Then R is an equivalence relation.

2. Let $A = \{1, 2, 3\}$. Find a relation on A such that the relation is:

   (a) reflexive, not symmetric, not transitive.

   (b) symmetric, not reflexive, not transitive.

   (c) transitive, not symmetric, not reflexive.

   (d) reflexive, symmetric, not transitive.

(e) reflexive, transitive, not symmetric.

(f) symmetric, transitive, not reflexive.

3. Let R be an equivalence relation on a nonempty set A, and let a and b be elements of A. Then $[a] = [b]$ if and only if $a \sim b$.

Partitions

DEFINITION 4. Let n be a positive integer, and let S be a set. A **partition** of S is a collection of subsets $S_1, S_2, \ldots, S_k$ such that each element of S lies in exactly one of these subsets. In other words,

$$S = S_1 \cup S_2 \cup \ldots \cup S_k \text{ and } S_i \cap S_j = \phi \text{for any } i \neq j.$$

THEOREM 2. *Let S be a set and $\sim$ be an equivalence relation on S. Then, the equivalence classes of $\sim$ form a partition of S.*

REMARK 3. Note that the converse is also true. Let $P = \{A_\alpha : \alpha \in I\}$ be a partition of a nonempty set A. Then there exists an equivalence relation R defined on A such that $P = \{[a] : a \in A\}$ is the set of equivalence classes determined by R.

(a) A relation R is defined on $\mathbb{Z}$ by xRy if $11x - 5y$ is even. Then R is an equivalence relation

i. Show R is an equivalence relation. Solution: You can show this.

ii. Determine the equivalence class of the relation R.

(b) Let $A = \mathbb{Z}$. Find four different partitions of the integers.