WORKSHEET 7

Date: 10/18/2021

Name:

Induction and its Brothers (or Sisters)

REMARK 1 (The First Principle of mathematical induction)**.** Suppose that for each positive integer n we have a statement $P(n)$. If we prove the following two things

1. $P(1)$ is true;
2. for all n, if $P(n)$ is true then $P(n+1)$ is also true;
   then $P(n)$ is true for all positive integers n.

REMARK 2 (The Second Principle of mathematical induction)**.** Let k be an integer. Suppose that for each integer $n \geq k$ we have a statement $P(n)$. If we prove the following two things

1. $P(k)$ is true;
2. for all $n \geq k$, if $P(n)$ is true then $P(n+1)$ is also true;
   then $P(n)$ is true for all positive integers $n \geq k$.

REMARK 3 (The Strong Form of Induction)**.** Suppose that for each integer $n \geq k$ we have statement $P(n)$. If we prove the following two things:

1. $P(k)$ is true; and
2. for all n, if $P(k), P(k+1), \ldots P(n)$ are all true, then $P(n+1)$ is true;
   then $P(n)$ is true for all $n \geq k$.

What is the distinction of the first principle of mathematical induction, the second principle of mathematical induction and strong induction? (You should be able to answer this question in Wednesday's section.)

Getting Our Hand's Dirty

1. Find a formula for $1+2+\ldots+n$ or in summation notation $\sum_{k=1}^{n} k$

2. Find a formula for $1 + 4 + 7 + \ldots + (3n - 2)$ for positive integers n, and then verify your formula by mathematical induction.

3. Prove that for every nonnegative integer n,

$$3|(2^{2n}-1).$$

4. Prove that for every nonnegative integer n, $(4^{3n} - 1)$ is a multiple of 9.

   Hint: Use induction.

Words of Caution

- I should inject a word of caution at this point, to wit, that one must be careful to establish both conditions of the remarks above before drawing any conclusions; neither is sufficient alone. Note, the validity of the induction step does not necessarily depend on the truth of the statement that one is endeavoring to prove. Take the false formula below:

  $$1+3+\ldots+(2n-1)=n^2+3.$$

  Assume that this statement holds for $n=k$; in other words,

  $$1+3+\ldots+(2k-1)=k^2+3.$$

  Knowing this, we then obtain

  $$1+3+\ldots+(2k-1)+(2k+1)=k^2+3+2k+1=(k+1)^2+3.$$

  But this is the exact statement we want to prove when $n=k+1$.

  What went wrong? Note that the base case was never verified; moreover, this statement is never true for any $n \in \mathbb{N}$.

- Guessing the answer. You can see that the induction only tells you how to prove some statements involving positive integers, such as some identities. However, it doesn't tell us how to find those identities. For example, consider the following identity

  $$\frac{1}{1\times 2}+\frac{1}{2\times 3}+\cdots+\frac{1}{n\times(n+1)}=\frac{n}{n+1}$$

  Certainly, this is a statement involving positive integers and you can prove this by induction. But the question is if I don't tell you the sum, how can you find it?

  The most naive way is that you just guess the answer. Since it is a statement about positive integers, you can always try some simple cases, like $n=1, n=2, n=3$, as long as you try enough cases, you will notice a patter. So you can guess the answer. How to verify your answer, you just prove if by induction. So, induction does not tell you the answer, but it can help you to verify your guessing.

5. Example: If $\{x_n\}$ is a sequence defined recursively by $x_1 = 1, x_2 = 3$, and $x_n = 2x_{n-1} - x_{n-2}$ for $n \geq 3$, then $x_n = 2n - 1$ for all natural numbers n.

6. Define a sequence $\{x_n\}$ recursively by $x_1 = 1, x_2 = 4$, and $x_n = 2x_{n-1} - x_{n-2} + 2$ for $n \geq 3$. Conjecture a formula for x_n and verify that your conjecture is correct. i.e prove the formula holds by induction.

7. Prove that

$$\sum_{k=1}^{n} \frac{1}{k^2} < 2.$$

Hint: Prove the stronger statement that the left hand side is less than $2 - \frac{1}{n}$.

8. Assume n is a non negative integer. Prove $3|(n^3 - n)$.

   (Hint: Just because induction is a choice doesn't always mean it is the easiest choice. Note that $n^3 - n$ factors to $(n-1)n(n+1)$. So, if you show that the product of three consecutive integers is a multiple of 3 then the statement above is proven. If you approach the problem this way you will need the following lemma: If $3|a$ and $k \in \mathbb{Z}$ then $3|ak$.)