Proofs For Worksheet 8
By The Students

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Preface

This pdf is a collection of proofs from our Worksheet 8. The students did a tremendous job in typing up the solutions and writing elegant proofs. I personally want to thank the scribes who learned LaTeXthis quarter. Learning a new computer language is a difficult task, especially when you have a million other things going on in your lives. Okay, that is enough praising. Below are the questions and proofs that I assigned on Wednesday. I also added the questionnaire- which I hope shares more light on how to attack a proof by induction. With all this being said, I hope this style of learning helped solidify your understand of a proof by indication. [If you find a typo please let me know and I will fix it.]

Chapter 1

Question 1: Dodgers

- What principle of induction are you going to use? Why did you choose this particular induction?
  We used the first remark of induction

- What is your induction hypothesis? Our induction hypothesis is that $(4^{3(k)} - 1)$ is a multiple of 9.

- What are you trying to prove?
  $(4^{3(k+1)} - 1)$ is a multiple of 9.

- Give a short paragraph of the strategy regarding your proof.

1. Prove that for every nonnegative integer n, $(4^{3n} - 1)$ is a multiple of 9.

   Hint: Use induction.

   Proof. We prove this statement by induction. First, we consider the base case ie $n = 1$. Let $t(n)$ be the statement $(4^{3n} - 1)$ is a multiple of 9. Then when $n = 1$, $(4^{3(1)} - 1)$ is 63 which is a multiple of 9, more explicitly it is 9(7). Assume our statement is true for some integer k ie: $(4^{3k} - 1)$ is a multiple of 9. So there exists an integer x such that $(4^{3k} - 1) = 9x$. We now need to show our statement is true for some $k + 1$. Consider the expression $(4^{3(k+1)} - 1)$. Simplifying a little gives us, $4^3 \cdot 4^{3k} - 1$. But by our induction hypothesis, we can write this as:

   $$4^3 \cdot (9x + 1) - 1 = 9(64x + 7).$$

   This shows our statement is true for $k + 1$. Hence if our statement is true for some integer k, then so is the $k + 1$ statement, and so it follows by induction that our statement is true for all non negative integers. □

Chapter 2

Question 2: Team #2

- What principle of induction are you going to use? Why did you choose this particular induction?

  We apply strong induction for problem number 2 because it gives us the most information for the problem since the statement holds for all values before k, and those values are present in the problem we need to prove.

- What is your induction hypothesis?

  Our induction hypothesis is that the statement $P(n) := x_n = 2n - 1$ holds true for values $n \geq 3$

- What are you trying to prove?

  We are trying to show or prove that $x_{n+1} = 2(n+1) - 1$ is true

- Give a short paragraph of the strategy regarding your proof.

  We first prove that the base case of x_3 and $P(3)$ is true, then we want to prove that the statement holds true for x_{n+1} while assuming that everything less than the base case will hold true because of Strong Induction. To do this, we will use algebra and the statements given to us.

2 If $\{x_n\}$ is a sequence defined recursively by $x_1 = 1, x_2 = 3$, and $x_n = 2x_{n-1} - x_{n-2}$ for $n \geq 3$, then $x_n = 2n - 1$ for all natural numbers n.

Proof:

It is given to us that x_n is a sequence with $x_1 = 1, x_2 = 3$, and $x_n = 2x_{n-1} - x_{n-2}$ for $n \geq 3$. We want to prove that statement $P(n)$: defined by $x_n = 2n - 1$ is true for all natural numbers n. We prove this by strong induction.
First, let us show the base case when $n = 3$. By using algebra and plugging into the given statement, we can find that $x_3 = 2x_{3-1} - x_{3-2} = 2x_2 - x_1$. Since we know the values for x_1 and x_2 because they are given to us, we can solve this using those values and plugging them in and get $2 \cdot 3 - 1 = 5$. This proves that the base case when $n = 3$ is true because plugging 3 in for $2 \cdot n - 1$, which equals 5. Now that we have proven the base case, let us consider x_{n+1}.
Assume that the statement holds true for the x_n and all $k \leq n$. Plugging $n + 1$ for the given statement we get

$$x_{n+1} = 2x_{(n+1)-1} - x_{(n+1)-2}.$$

By simplifying we get the statement $x_{n+1} = 2x_n - x_{n-1}$. We can split this statement up into two parts: the first part being P(n) and the second part after the subtraction sign being P(n-1). Let us plug in for $P(n) = 2n - 1$ and $P(n-1) = 2(n-1) - 1$. Using algebra we get $x_{n+1} = 2(2n-1) - (2(n-1) - 1)$, which we can simplify into $4n - 2 - (2n - 3)$. We can simplify even further and get $4n - 2 - 2n + 3$, and if we combine like terms we get the final statement of $x_{n+1} = 2n + 1$. This is the same thing as P(n+1) because if we rearrange this statement by adding and subtracting values, we get the statement $2n + 2 - 1$, and if we factor we can get $2(n+1) - 1$. This proves that $x_n = 2n - 1$ for all natural numbers n, including n+1. So by the Principle of Mathematical (Strong) Induction, our statement P(n) holds for all natural numbers n.

Chapter 3

Question 3: Number One

3) Define a sequence $\{x_n\}$ recursively by $x_1 = 1$, $x_2 = 4$, and $x_n = 2x_{n-1} - x_{n-2} + 2$ for $n \geq 3$. Conjecture a formula for x_n and verify that your conjecture is correct. i.e prove the formula holds by induction.

Proof. We claim x_n can be expressed as n^2 for n in the natural numbers. The proof is by strong induction. Assume our statement $P(n)$ is defined to be $x_n = n^2$. First, the base case. Let $n = 3$. Then $n^2 = 9$ and x_3 is also 9 by the recursive relation of x_n. Assume our statement $P(k)$ is true for all integers $k \leq n$ i.e
$n^2 = x_k = 2x_{k-1} - x_{k-2} + 2$. We want to show that $P(k+1)$ is true. Since $x_k = k^2$, then $x_{(k-1)} = (k-1)^2$. Now,

$$x_{(k+1)} = 2x_{(k-1)+1} - x_{(k-2)+1} + 2$$
$$= 2x_k - x_{k-1} + 2.$$

Next, we can plug in x_k into the equation above,

$$x_{(k+1)} = 2k^2 - (k-1)^2 + 2.$$

We can then distribute $(k-1)^2$ giving us,

$$x_{(k+1)} = 2k^2 - k^2 - 2k + 1 + 2$$
$$= k^2 - 2k + 3.$$

Lastly, we can factor the equation into,

$$x_{(k+1)} = (k+1)^2.$$

Hence, $P(k+1)$ is true. By the principle of mathematical induction, our statement is true for all integers n. □

Chapter 4

Question 4: giants

- What principle of induction are you going to use? Why did you choose this particular induction? We used Remark 1 of the Principle of Mathematical Induction because we felt it was the one we understood the most.

- What is your induction hypothesis?
  $P(n) = \sum_{k=1}^{n} \frac{1}{k^2} < 2 - \frac{1}{n}$

- What are you trying to prove?
  We are trying to prove that $P(1)$ is true and that if $P(n)$ is true for some integer, then so is $P(n+1)$

- Give a short paragraph of the strategy regarding your proof.
  Our strategy regarding this proof was to first show that the statement is true when $n = 1$. Then, by assuming that $P(n)$ is true, we will show that $P(n+1)$ is also true.

First, we define $\sum = \text{🎃}$. Prove that

$$\text{🎃}_{k=1}^{n} \frac{1}{k^2} < 2.$$

Hint: Prove the stronger statement that the left hand side is less than $2 - \frac{1}{n}$.

Proof:

We prove this by induction. Let $P(n)$ be the stronger statement $\text{🎃}_{k=1}^{n} \frac{1}{k^2} \leq 2 - \frac{1}{n}$. We first show that $P(1)$ is true. We have $\frac{1}{1^2} \leq 2 - \frac{1}{1} \Rightarrow 1 \leq 1$. Suppose now that our inductive hypothesis is that $P(n)$ is true for some integer n (i.e.$\text{🎃}_{k=1}^{n} \frac{1}{k^2} \leq 2 - \frac{1}{n}$). Adding $\frac{1}{(n+1)^2}$ to both sides, we get $\text{🎃}_{k=1}^{n+1} \frac{1}{k^2} < 2 - \frac{1}{n} + \frac{1}{(n+1)^2} = 2 - \frac{(n+1)^2 - n}{n(n+1)^2}$. We then have $2 - \frac{n^2+n+1}{n(n+1)^2} < 2 - \frac{n(n+1)}{n(n+1)^2} = 2 - \frac{1}{n+1}$.This follows from the fact that $n^2 + n + 1 > n^2 + n$ and multiplying by a negative will change the sign. Thus, we have $\text{🎃}_{k=1}^{n+1} \frac{1}{k^2} \leq 2 - \frac{1}{n+1}$. $P(n+1)$ holds and the result follows by induction. Now, it is always the fact for any positive integer n, $2 - \frac{1}{n} < 2$. Hence, the statement we first wished to show follows immediately from the stronger statement we proved.

Chapter 5

Question 5: PCV Flutes

- The Statement: Assume n is a non negative integer. Prove $3 \mid (n^3 - n)$.
- What principle of induction are you going to use? Why did you choose this particular induction?
  We are going to use the second principle of induction which we chose because we understood it best.
- What is your induction hypothesis?
  Our induction hypothesis is that $P(k)$ is true which is $3 \mid (k^3 - k)$.
- What are you trying to prove? We are trying to prove that $P(k)$ implies $P(k+1)$. i,e assuming $3 \mid (k^3 - k)$ is true, then $3 \mid ((k+1)^3 - (k+1)$ is true.
- Give a short paragraph of the strategy regarding your proof.
  We want to show that $P(k)$ implies $P(k+1)$ by finding how $P(k+1)$ is a multiple of 3.

5 Assume n is a non negative integer. Prove $3 \mid (n^3 - n)$.

Proof. The proof is by induction. Let $P(n)$ denote the statement "$3 \mid (n^3 - n)$". We first show that $P(1)$ is true. We have $1^3 - 1 = 0$ which is a multiple of 3 so this implies that $P(1)$ is true. Suppose now as our inductive hypothesis that $P(k)$ is true, ie. that $3 \mid (k^3 - k)$. Then, we can say that $3a = k^3 - k$ for some $a \in \mathbb{Z}$. This also means $k^3 = 3a + k$. Consider $(k+1)^3 - (k+1)$. This simplifies to $k^3 + 3k^2 + 2k$. From our inductive hypothesis, we know that $k^3 = 3a + k$ so we can change our expression to $3a + k + 3k^2 + 2k$; since $(a + k^2 + k) \in \mathbb{Z}$. This simplifies to $3(a + k^2 + k)$ which $(k+1)^3 - (k+1)$is a multiple of 3 i.e $3 \mid ((k+1)^3 - (k+1)$ Hence, $P(k+1)$ holds, and so the results follows by induction. $\square$

Chapter 6

Question 5: Team #3

Assume n is a non negative integer. Prove $3|(n^3 - n)$.

(Hint: Just because induction is a choice doesn't always mean it is the easiest choice. Note that $n^3 - n$ factors to $(n-1)n(n+1)$. So, if you show that the product of three consecutive integers is a multiple of 3 then the statement above is proven. If you approach the problem this way you will need the following
lemma: If $3|a$ and $k \in \mathbb{Z}$ then $3|ak$.)

Proof. Let $P(n)$ denote the statement "n is a non negative integer, $3|(n^3 - n)$". We prove this statement by induction. We first check that $P(1)$ is true. $1^3 - 1 = 0$, which is a multiple of 3. Suppose that $P(k)$ is true for some integer k, then $3y = k^3 - k$, $y \in \mathbb{Z}$. consider the expression $(k+1)^3 - (k+1)$. We simply the expression, and this gives the following:
$(k^2 + 2k + 1)(k+1) - (k+1)$
$= (k^2 + 2k)(k+1)$
$= k^3 + 2k^2 + k^2 + 2k$
$= k^3 + 3k^2 + 2k$
$= k^3 - k + 3k^2 + 3k$
$= 3y + 3(k^2 + k)$
$= 3(y + k^2 + k)$. We note that $y + k^2 + k$ is an integer since each term is an integer. Hence, 3 divides $(k+1)^3 - (k+1)$. Therefore, $P(k+1)$ is true, and so the result follows by induction.

□