Worksheet 2

Date: 9/29/2021

Name:

Producing new statements from old

For two given statements P and Q, a common way to produce a new statement from them is by inserting the word ”or” or ”and”. The **disjunction** of the statements P and Q is the statement :

P **or** Q.

We usually denote this by $P \vee Q$

P	Q	$P \vee Q$
T	T	
T	F	
F	T	
F	F	

The **conjunction** of the statements P and Q is the statement :

P **and** Q.

We usually denote this by $P \wedge Q$

P	Q	$P \wedge Q$
T	T	
T	F	
F	T	
F	F	

The statement formed from two given statements in which we will be most interested is the **implication** (also called the **conditional**). For statements P and Q, the implication is the statement:

If P, **Then** Q.

This is denoted by $P \Rightarrow Q$.

P	Q	$P \Rightarrow Q$
T	T	**T**
T	F	**F**
F	T	**T**
F	F	**T**

We will explain this truth table on the next couple of pages.

The **negation** of a statement P is the statement:

not P

and is denoted by $\sim P$.

P	$\sim P$
T	
F	

- The **converse** of $P \Rightarrow Q$ is $Q \Rightarrow P$.
- The **inverse** of $P \Rightarrow Q$ is the implication $\sim P \Rightarrow \sim Q$.
- The **contrapositive** of $P \Rightarrow Q$ is the implication $\sim Q \Rightarrow \sim P$.

  (Note that this is logically equivalent to the original conditional statement.Show this via a truth table. The truth of the converse and inverse need not be dependent of the original conditional statement.)

Explanation for the implication truth table

A student is taking a math class(let's say this one) and is currently receiving a B+. He visits his instructor a few days before the final examination and asks her, " Is there any chance that I can get an A in this course?" His instructor looks through her grade book and says," If you earn an A on the final exam, then you will receive an A for your final grade". We now check the truth or falseness of this implication based on the various combinations of truth values of the statements:

$$P: \text{ You earn an A on the final exam.}$$

and

$$Q: \text{ You receive an A for your final grade.}$$

Which make up the implication.

Suppose first that P and Q are both true. That is, the student receives an A on his final exam and later learns he got an A for his final grade in the course. Did his instructor tell the truth?

Second, suppose that P is true and Q is false. So the student got an A on his final exam but did not receive an A as a final grade, say he received a B. Certainly, his instructor did not do as she promised(as she will soon be reminded by her student). Did his instructor tell the truth?

Third, suppose that P is false and Q is true. In this case, the student did not get an A on his final exam, but when he received his final grades, he learned that his final grade was an A. In this case, the instructor did not lie; so she told the truth.

Finally, suppose that P and Q are both false. That is, suppose the student did not get an A on his final exam, and he also did not get an A for a final grade. The instructor did not lie here either. She only promised the student an A *if* he got an A on the final exam. She promised nothing if the student did not get an A on the final exam. So the instructor told the truth.

Practice Problems

1. Prove the empty set is unique.

2. Let A and B be sets. If $B \subseteq A$, then $A \cup B = A$.

3. Let $x \in \mathbb{Z}$. If x is even, then x^2 is even.

   Prove the converse of the statement.

   Hint: use the contrapositive of the statement.

Negation of conditional statement

Logically equivalent form of an implication statement.

P	Q	$\sim P$	$P \Rightarrow Q$	$(\sim P) \vee Q$
T	T			
T	F			
F	T			
F	F			

What is the negation of the implication statement?

P	Q	$\sim Q$	$\sim (P \Rightarrow Q)$	$P \wedge (\sim Q)$
T	T			
T	F			
F	T			
F	F			

Prove De'Morgan's Law : $\sim (P \vee Q)$ is logically equivalent to $(\sim P) \wedge (\sim Q)$

P	Q	$\sim P$	$\sim Q$	$P \vee Q$	$\sim (P \vee Q)$	$(\sim P) \wedge (\sim Q)$
T	T					
T	F					
F	T					
F	F					

and : $\sim (P \wedge Q)$ is logically equivalent to $(\sim P) \vee (\sim Q)$

P	Q	$\sim P$	$\sim Q$	$P \wedge Q$	$\sim (P \wedge Q)$	$(\sim P) \vee (\sim Q)$
T	T					
T	F					
F	T					
F	F					

Negation

- We write $\exists$ to mean "there exists". Example: $\exists x \in \mathbb{Z} \ni x+1=0$ reads "there exists x in the integers such that $x+1=0$.
- We write $\forall$ to mean "for all". Example: $\forall x \in \mathbb{Z}$, $-x \in \mathbb{Z}$ reads " for all x in the integers, $-x$ is in the integers".
- Whats the negation of the statements above?
- Negation of $\leq$ is $>$

1. Negate the following statements:

   (a) $\forall n \in \mathbb{Z} \ni n$ is prime, n is odd.

   (b) $\exists x, y \in \mathbb{Z}$ such that $x+y \notin \mathbb{Z}$.

   (c) $\forall x \in \{y \mid y \in \mathbb{Z}, y \geq 1\}$, $\exists z \in \mathbb{Z} \ni 5x^2+5z+1$ is prime.

   (d) $\exists x \in \mathbb{Z}$ such that $\forall n \in \mathbb{Z} x \neq n^2+2$

   (e) Let A and B be non empty sets. What is $\sim (A \subseteq B)$. Recall the definition of a subset:

   $$\forall x \in A, x \in A \Rightarrow x \in B.$$

Practice Problems

1. Prove $\log 2$ is irrational

2. Let a, b be rationals and x be irrational. Show that if

$$\frac{x+a}{x+b} \text{ is rational,}$$

   then $a = b$.

3. Consider a rectangle with positive sides $a, b \in \mathbb{R}$. Is it possible to find values a, b such that the perimeter is rational but the area is irrational?