# HOMEWORK 5

Date: 2/11/21

Name:

1. Compute the principle value of the arguments of the following complex numbers:
   $1+i, \frac{1}{2}+i\frac{\sqrt{3}}{2}, (1+i)^3, (\frac{1}{2}+i\frac{\sqrt{3}}{2})^{243}, (1+i)^2(\frac{1}{2}+i\frac{\sqrt{3}}{2})^3$

2. Let $\arg z$ denote the principle value of the argument for $z \neq 0$, that is, $-\pi < \arg z \leq \pi$. For real, $x, y$ with $x < 0$ show:

   (i) $\lim_{y\to 0} \arg(x + i|y|) = \pi$

   (ii) $\lim_{y\to 0} \arg(x - i|y|) = -\pi$

   Compute the corresponding limits when $x = 0$ and when $x > 0$.

3. Compute the following principal logarithms: $Log(3i), Log(-2i), Log(1+i), Log(-1), Log(z^{10})$ where $z = 2e^{i\pi/3}, Log(x)$ for real $x \neq 0$.

4. For $z_1, z_2 \neq 0$ show that

   $$Log(z_1 z_2) = Log(z_1) + Log(z_2) + 2n\pi i$$

   where $n$ is an integer that need not be zero. Specify the value that $n$ may take. Show that a logarithm of $z_1 z_2$ is of the form

   $$\log(z_1) + \log(z_2)$$

   provided that appropriate values of the logarithms are taken.

5. Let $n$ be a positive integer. A complex number $\omega$ is said to be an nth roof of unity if $\omega^n = 1$.

   (i) Find all nth roots of unity in polar coordinates and draw a picture.

   (ii) For $n = 2, 3, 4$ express the nth roots of unity in the form $x + iy$.

   (iii) If $\omega_1, \omega_2$ are nth roots of unity, show that the following are also nth roots of unity:

   $$\omega_1^n \qquad \omega_1\omega_2 \qquad \omega_1/\omega_2$$

   (iv) For given $r, \theta \in \mathbb{R} (r > 0)$, find all $z \in \mathbb{C}$ such that

   $$z^n = re^{i\theta}$$

   (v) If $z_1^n = z_2^n$, show that $z_1 = \omega z_2$ where $\omega$ is an nth root of unity.

6. For $z, \beta \in \mathbb{C}, z \neq 0$, define the principal value of $z^\beta$ to be

   $$z^\beta = e^{\beta Log(z)}$$

   Compute the principal values of the following powers:

   $$1^{\sqrt{2}} \qquad (-2)^{\sqrt{2}} \qquad i^i \qquad 2^i \qquad (3-4i)^{1+i} \qquad (3+4i)^5$$

   $$(-2)^{\sqrt{2}} = e^{\sqrt{2}(\ln(2)+i\pi} = e^{\sqrt{2}\ln(2)} e^{(i\sqrt{2}\pi)}$$

   Remember that we are considering the principle branch. So our angle needs to be:

   $$\sqrt{2}\pi - 2\pi$$

7. Draw the following paths and specify all the continuous choices of arguments along them:

(i) $\gamma(t) = 2e^{-it}(t \in [0, 4\pi])$

(ii) $\gamma(t) = t + i(1-t)(t \in [0, 1]$

This is a straight line path from $i$ to 1. The claim is the following:

$$\theta(t) = \begin{cases} \arctan\left(\frac{1-t}{t}\right) & , t \in (0, 1] \\ \frac{\pi}{2} & , t = 0. \end{cases}$$

Now check that $e^{i\theta(t)}$ satisfies our criteria. When $t \in (0, 1]$ we get

$$e^{i\arctan\left(\frac{1-t}{t}\right)} = \cos\left(\arctan\left(\frac{1-t}{t}\right)\right) + i\sin\left(\arctan\left(\frac{1-t}{t}\right)\right)$$

Verify by using the facts about trigonometry

(iii) $\gamma(t) = t - 1 + it^2(t \in [-1, 1])$

$$\theta(t) = \begin{cases} \pi + \arctan\left(\frac{t^2}{t-1}\right) & , t \in [-1, 1) \\ \frac{\pi}{2} & , t = 1. \end{cases}$$

(iv) $\gamma(t) = \begin{cases} t + i(1-t) & \text{if } t \in [0, 1] \\ 1 + i(1-t) & \text{if } t \in [1, 2] \end{cases}$

In each case compute the winding number of the path round the origin.