HOMEWORK 6

Date: 2/11/21

Name:

1. Find the winding number $w(\gamma, z_0)$ for each of the following choices of γ, z_0:

   (i) $\gamma(t) = 2e^{-it}(t \in [0, 2\pi])$; $\quad z_0 = 1, 3i$

   (ii) $\gamma(t) = t + i(1-t)(t \in [0,1])$; $\quad z_0 = 1+i, -i, 10i$

   I will only do the case when $z_0 = -i$ since the rest follow by the same method. Let $\Gamma(t) = \gamma - z_0 = t + i(2-t)$ where $t \in (0,1)$. Then $\omega(\gamma, z_0) = \omega(\Gamma, 0)$. Define the continuous argument of Γ as

   $$\theta(t) = \begin{cases} \pi/2 & \text{if } t = 0, \\ \arctan(\frac{2-t}{t}) & \text{if } t \in (0,1]. \end{cases}$$

   Now use the definition of the winding number of a path to show $\omega(\Gamma, 0) = -1/8$.

2. For each of the following, draw a picture of the path and use a sensible method to compute $\int_\gamma 1/(z-z_0)dz$:

   (i) $\gamma(t) = te^{-it}(t \in [\pi, 5\pi]) \quad z_0 = 0.$

   (ii) $\gamma(t) = -it(t \in [0,1]) \quad z_0 = 1$
   $A = \int_\gamma \frac{1}{z-z_0}dz = \int_0^1 \frac{t}{t^2+1}dt + i\int_0^1 \frac{1}{t^2+1}dt$ We make a quick note. From here we have three different ways of solving it. The easiest way is just to compute the integral with good old calculus. The second method is to use the fact that $\omega(\gamma,1) = \frac{1}{2\pi}im(A)$. Professor Agboola mentioned this in his class notes. This is probably why some people wanted to use the fact about winding numbers. The third method is to find the antiderivative of $1/z - z_0$. If you use this method **YOU NEED TO SPECIFY A BRANCH CUT**. The principal branch will work in this case.

   (iii) $\gamma(t) = it(t \in [-1,1]) \quad z_0 = 1$

   (iv) $\gamma(t) = \sigma + [1,2] + \rho + [-2,-1] \quad z_0 = 0$
   where $\sigma(t) = e^{i(\pi-t)}, (t \in [0,\pi])$ and $\rho(t) = 2e^{-it}(t \in [0,4\pi])$.
   I believe we have a typo in this question. To make this question more sensible ρ should be from $[0,3\pi]$. Or maybe $[-2,-1]$ should be from $[2,-1]$. Either way, I think it wanted the path to be closed. Recall that "addition of paths" means the end points of the first path agrees with the initial point of our second path.

3. Compute (by eye) the winding number of the given closed paths round a point in each of the connected components A,B,C,… of the complement, drawn below.

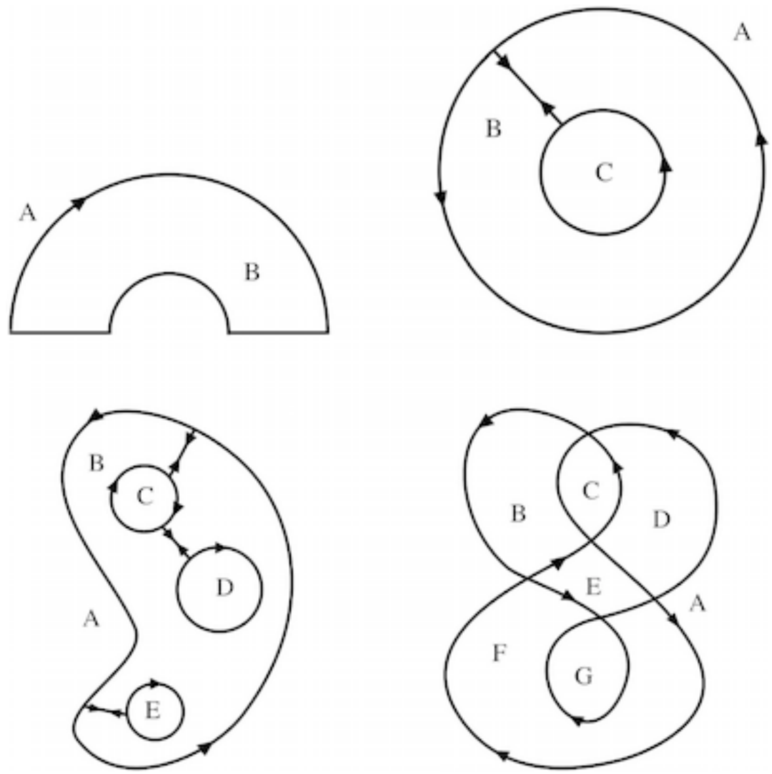

Figure 7.11 Compute (by eye) the winding numbers of these closed paths round a point in each component A, B, C , ...

4. State which of the following are star domains, proving the existence of a star centre for those that are, and justifying your answer for those that are not.

   (i) $\{z \in \mathbb{C} : z \neq x + 0i \text{ where } |x| \geq 1\}$ This is a star domain. The origin is the center of the star. WLOG assume w is in our set and in the first quadrant.
   Case 1: $w = a + bi$ where $a, b \in \mathbb{R}$ and $a, b > 0$. Then $[0, w] = tw$ where $t \in [0, 1]$. Now $wt = at + ibt$. For our path not to be in our domain there must exist a $t \in (0, 1]$ such that $bt = 0$. But t and b are real numbers. And if there product is zero b or t must be zero. But this is not the case since we assumed each are positive real numbers. Case 2: consider now the case when $0 < a < 1$ and $b = 0$. This is left for you.

   (ii) $\{z \in \mathbb{C} : |z| > 1\}$ This is not a star domain. Take any point z in our set and consider $-z$. If we consider the path $[z, -z]$, there exist a $t \in (0, 1)$ such that our path passes through the origin which is not in our set. Note: to guarantee full marks you need to explicate find this t. With a moments thought it becomes apparent what t should be.

   (iii) $\{z \in \mathbb{C} : z \neq e^{it} \text{ for } t \in [0, \pi]\}$

   (iv) $\{z \in \mathbb{C} : |z| > 1 \text{ and either } im\, z > 0 \text{ or } re\, z > 0\}$

5. Let $D = \mathbb{C} \setminus \{0\}$. For $z_0 \in D$, specify a local antiderivative in some neighbourhood of z_0 for each of the following functions:

   (i) $1/z$

   (ii) $1/z^2$

   (iii) $(z+1)/z^2$

   (iv) $(\cos(z)/z$

   (v) $(\sin(z)/z)$