HOMEWORK 5

Date: 2/25/21

Name:

8.8.3 Let

$$\begin{aligned}\gamma_1(t) &= -1+\frac{1}{2}e^{it}(t\in[0,2\pi]) \\ \gamma_2(t) &= 1+\frac{1}{2}e^{it}(t\in[0,2\pi]) \\ \gamma(t) &= 2e^{-it}(t\in[0,2\pi]\end{aligned} \tag{1}$$

If $f(z)=1/(z^2-1)$ use theorem 8.9 to deduce that

$$\int_\gamma f = \int_{\gamma_1} f + \int_{\gamma_2} f$$

Interpret this statement in terms of the winding numbers of γ,γ_1,γ_2 round $1,-1$.

First, find the winding number of the compliment of our region. So $z=\pm1$. Show that the sum of the winding number is zero. By the theorem you get

$$\int_{\gamma_1} f(z)dz+\int_{\gamma_2} f(z)dz+\int_{-\gamma} f(z)dz=0$$

Use the partial fraction technique to show

$$\frac{1}{z^2-1}=\frac{1}{2(z-1)}-\frac{1}{2(z+1)}$$

Now, this tells you about the winding. In summary, the winding numbers of γ at -1 and 1 is equal to the sum of the difference between the winding numbers of γ_1 and γ_2 about -1,1.

8.8.4 Show that $D = \{z \in \mathbb{C} : z \neq \pm 1\}$ is not simply connected. Let

$$\begin{aligned} L_1 &= \{x + iy \in \mathbb{C} : y = 0, x \leq -1\} \\ L_2 &= \{x + iy \in \mathbb{C} : y = 0, x \geq 1\} \\ D_0 &= D \setminus \{L_1 \cup L_2\} \end{aligned} \tag{2}$$

Show that D_0 is simply connected. Is it a star domain? Does $f(z) = 1/(z^2 - 1)$ have an antiderivative in D_0? In each case justify your answer.

Show that the contour of a disc of radius 1 around 1 is not equal to zero when $f = 1/(z^2 - 1)$. Use theorem 8.12 to show it is not simply connected. D_0 is a star domain, this was a previous homework problem. Use Corollary 8.4 to show and theorem 8.12 to show D_0 is simply connected. What theorem now tells you that f has an antiderivative in D_0? or you can construct the antiderivative explicitly.

8.8.6 Let $D = \{z \in \mathbb{C} : z \neq \pm i\}$ and let γ be a closed contour in D.

Find all the possible values of $\int_\gamma 1/(z^2+1)dz$. If σ is a contour from 0 to 1, find all possible values of $\int_\sigma 1/(z^2+1)dz$.

The way I answered this question in section is not very efficient. It is correct but not what the author wanted. Use partial fractions on our function $1/(z^2+1)$. You will see that we get the difference of the winding numbers. The same answer as we got in section.

Consider the contour $[0,1]$ and the following contour: $\{i + e^{it},\ (t \in [-\pi/2, 3\pi/2]\} + [0,1]$. Are these two contours the same? Most students put that our path is independent but that is not the case. If it was independent then any integral of a closed contour must be zero, but that is not the case. Some students also said we have an antiderivative in our domain but that is also not the case. A local anti yes but we are looking for a global one in our domain.

8.8.7 Let $\gamma_1 = S_1 + L - S_2 - L, \gamma_2 = S_1 + L + S_2 - L$, where

$$\begin{aligned} S_1(t) &= e^{it}(t \in [0, 2\pi]) \\ S_2(t) &= 2e^{it}(t \in [0, 2\pi]) \\ L &= [1, 2] \end{aligned} \tag{3}$$

Describe the inside and outside of γ_1 and γ_2. Let $f(z) = \cos(z)/z$. By writing $\cos(z)$ as a power series and considering $f(z) = 1/z + g(z)$, or otherwise, compute $\int_{\gamma_1} f$ and $\int_{\gamma_2} f$. Compare the results with theorem 8.10.

This one is straight forward so nothing needs to be said.

8.8.8 Let $D = \mathbb{C} \setminus \{z_1, \ldots, z_k\}$ where $z_j \in \mathbb{C}$, and suppose that f is differentiable in D. Show that for any closed contour in D,

$$\int_\gamma f = \sum_{r=1}^{k} n_r \int_{S_r} f$$

where S_r is a sufficiently small circle centre z_r and n_r is an integer. If $\lim_{z \to z_r} f(z) = a_r \in \mathbb{C}$ for $r = 1, \ldots k$ show that

$$\int_\gamma f = \sum_{r=1}^{k} 2\pi i n_r a_r$$

For the first part you will need induction. To use the induction step you need to split the contour into two contours where "cancellation of paths" is used. For the second part use the first part and then induction. We went over this in section so I don't plan to write an entire solution. The key fact is the following

$$2\pi i = \int_{S_r} 1/(z - z_r) dz$$

10.9.1 Find the Taylor series at 0 of $f(z) = Log(1+z)$, where Log is the principal value. What is the disc of convergence? Answer the same questions for

$$g(z) = e^{\alpha Log(1+z)} \text{ where } \alpha \in \mathbb{C}$$

use the fact that $\frac{1}{1+z} = \sum(-z)^n$ is valid for $|z| < 1$. Integrate this and explain why its valid to do so and what the radius of convergence of the new series is. For the second part, us the definition of the Taylor series to find each coefficients. Why is our radius of convergence 1?

10.9.3 Taylor expand the following functions around 0, and find the radius of convergence.

Solutions: note that we can find the coefficients of our Taylor series by the definition, or we can use the fact that a Taylor series is unique. I will use the latter method.

(i) $\sin^2(z)$

Use the fact that $\sin^2(z) = \frac{1-\cos(2z)}{2}$ and expand this series. The radius of convergence is the whole complex plane.

(ii) $z^2(z+2)^{-2}$

Write $\frac{1}{z+2}$ as $\frac{1}{2(1+z/2)} = \frac{1}{2}\sum(-1)^n(z/2)^n = g(z)$. This expansion is valid when $|z| < 2$. Differentiate $g(z)$ and multiply by z^2. Why does this new power series have radius of convergence $|z| < 2$?

(iii) $(az+b)^{-1}, (a,b \in \mathbb{C}, b \neq 0)$

$(az+b)^{-1} = \frac{1}{b(1+az/b)}$. Expand like we did in *ii*. You can find the radius of convergence of this.

(iv) $\int_0^z e^{w^2} dw$

$e^{w^2} = \sum \frac{(w^2)^n}{n!}$ which is valid in the whole complex plane. Now, why can you swap the integral and sum? Answer: we know that our series converges uniformly to our function in our radius of convergence. This is an important fact.

(v)

$$\begin{cases} \sin(z)/z & ,z \neq 0 \\ 1 & ,z = 0. \end{cases}$$

Our function is differentiable at $z = 0$ which means we have a power series centered at $z = 0$. Expand the series for $\sin(z)/z$. From the uniqueness of the Taylor series we have that the series our series is the Taylor series of $\sin(z)/z$. Why is the radius of convergence the whole complex plane?

(vi) $\int_0^z \sin(w)/w dw$

Use the expansion from *v* and integrate.

10.9.4 Define the numbers c_n by the Taylor series

$$\sec(z) = \sum_{n=0}^{\infty} (-1)^n \frac{c_{2n}}{(2n)!} z^{2n}$$

Prove that

$$\begin{aligned} c_0 &= 1 \\ 0 &= c_0 + c_2 \binom{2n}{2} + c_4 \binom{2n}{4} + \ldots + c_{2n} \binom{2n}{2n} \end{aligned} \tag{4}$$

Show that c_{2n} is always an integer and calculate it for $n \leq 5$.

Multiply both sides by $\cos(z)$ and expand $\cos(z)$ in terms of its power series. On page 71 of our book it tells you how to find the coefficients for the product of two power series. The equation in (4) follows from this. Use the second principle of mathematical induction to show the coefficients are always integers. You can calculate the first couple of c_{2n} by the definition of Taylor series or using (4).

10.9.5 Let

$$(1-z-z^2)^{-1} = \sum F_n z^n$$

Prove that

$$F_0 = F_1 = 1 \qquad F_n = F_{n-1} + F_{n-2} \ (n \geq 2)$$

This is the recursive definition of the *Fibonacci numbers* $1,1,2,3,5,8,13\dots$ By expanding $(1-z-z^2)^{-1}$ in partial fractions, prove that

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right]$$

This one is straight forward so not much is needed.

10.9.8 Let $f(z)$ have Taylor series $\sum a_n z^n$ for $|z| < R$. Let $\omega = e^{2\pi i/3}$ and define

$$g(z) = \frac{1}{3}(f(z) + f(\omega z) + f(\omega^2 z))$$

Show that

$$g(z) = \sum a_{3n} z^{3n}$$

for $|z| < R$. Find similar expressions for $\sum a_{3n+1} z^{3n+1}$ and $\sum a_{3n+2} z^{3n+2}$. (Hint: $1 + \omega + \omega^2 = 0$)

Find the Taylor series for each $(f(z), f(\omega z), f(\omega^2 z))$ combine the terms to get g and use the hint. Why does $f(\omega z), f(\omega^2 z))$ have radius of convergence R?

$$\sum b_k z^k = \frac{1}{3}(f(z) + \omega^2 f(\omega z) + \omega f(\omega^2 z))$$

$$\sum c_k z^k = frac13(f(z) + \omega f(\omega z) + \omega^2 f(\omega^2 z))$$

Match the corresponding series to the expressions above.