MATH 122A HOMEWORK 3

Due date: 1/28/21

Name:

3.6.6 Find the radius of convergence of the following series:

(i) $\sum z^n/n$

(ii) $\sum z^n/n!$

(iii) $\sum n!z^n$

(iv) $\sum z^n n^k$ where k is a positive integer.

(v) $\sum z^{n!}$

3.6.8 Find the radius of convergence of the following series:

(i) $z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \dots$

(ii) $1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \dots$

(iii) $z - \frac{z^2}{2!} + \frac{z^3}{3!} - \frac{z^4}{4!} + \dots$

(iv) $1 - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \dots$

(v) $1 + az + \frac{a(a-1)}{2!} z^2 + \dots + \frac{a(a-1)\cdots(a-n+1)}{n!} z^n + \dots$

(Note that in part (iv) the radius of convergence may differ for different values of a.)

3.6.12 Prove that if each of the series $\sum a_n z^n, \sum b_n z^n$ and $\sum a_n b_n z^n$ has radius of convergence equal to 1, then the series $\sum a_n b_n^2 z^n$ and $\sum a_n^2 b_n z^n$.

Common mistake: Remember that we are give that the radius of convergence of $\sum a_n z^n$ is 1. This means

$$\frac{1}{\limsup_{n\to\infty} |a_n|^{1/n}} = 1.$$

This tells us nothing about the ratio test for a_n, i.e.

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|.$$

Recall an important fact from analysis: assume a_n and b_n are positive real sequences, then

$$\limsup a_n b_n \leq \limsup a_n \limsup b_n.$$

With this fact you can show that 1 is a lower bound for the radius of convergence. Now, how can you bound the radius of convergence from above? You should use the given some how. Notice that we are not guaranteed that we can distribute the limit into each sequence as we did in the previous homework set. We were allowed to do it before because we had conditions which made the limit nice enough.

3.6.15 Suppose that the power series $\sum_{n=0}^{\infty} a_n z^n$ has a recurring sequence of coefficients: that is, $a_{n+k} = a_n$ for some fixed positive integer k and all n. Prove that the series converges for $|z| < 1$ to a rational function $p(z)/q(z)$ where p, q are polynomials, and that the roots of q are all on the unit circle. What happens if $a_{n+k} = a_n/k$?

4.7.2 Show that $f(z) = |z|$ is continuous everywhere and differentiable nowhere. Show that $f(z) = |z|^2$ is continuous everywhere and differentiable at the origin but nowhere else.

4.7.4 Let

$$f_n(z) = (1 + \frac{z}{n})^n$$

Show that

$$f_n'(z) = f_{n-1}\left(\frac{(n-1)z}{n}\right)$$

What do you notice as $n \to \infty$?

4.7.6 Let $f(z)$ be a polynomial in $z \in \mathbb{C}$. Prove that the function $g(z) = \overline{f(\bar{z})}$ is differentiable everywhere, but that $h(z) = \overline{f(z)}$ is differentiable at 0 if and only if $f'(0) = 0$.

We give a couple of proofs for this result. We also prove a more general statement. The first proof might be what the book intended to be honest. The second proof holds for entire functions, with a slight condition. The third proof holds for certain domains with the proof identical to two. Lastly, we give a power series proof.

Proof. Assume f is a polynomial. Then there exists a natural number n such that

$$f(z) = \sum_{j=0}^{n} a_j z^j.$$

By the properties of conjugation

$$g(z) = \overline{f(\bar{z})} = \sum_{j=0}^{n} \overline{a_j} z^j$$

$g(z)$ is a polynomial with complex coefficients. Hence, g is differentiable; moreover, its derivative is

$$g'(z) = \sum_{j=0}^{n-1} j\overline{(a_l)} z^j$$

$\square$

Assume f is an entire function and each component of f has continuous partial derivative. Then $g(z) = \overline{f(\bar{z})}$ is differentiable everywhere

Proof. Assume f is entire. Then there exist $u(x,y)$ and $v(x,y)$ such that $f(x+iy) = u(x,y) + iv(x,y)$ whose partials exist, are continuous, and satisfy the Cauchy Riemann equations. Then

$$g(z) = u(x,-y) - iv(x,-y).$$

Show that the Cauchy-Riemann equations for g hold each component of g yields continuous partial derivatives with the partial derivatives also existing. All of this should follow from functions u and v and our assumption. The proof of the main statement now follows since a polynomial satisfy all these conditions. $\square$

Proof. Let A be an open subset of $\mathbb{C}$. Define $B = \{z | \bar{z} \in A\}$. Assume f is an analytic on A and each component of f has continuous partial derivative. Then $g(z) = \overline{f(\bar{z})}$ is analytic on B. Follow the same proof as the previous one. Now when our function is a polynomial the result holds. $\square$

Proof. Assume f is a polynomial. Then there exists a natural number n such that

$$f(z) = \sum_{j=0}^{n} a_j z^j = \sum_{k=0}^{\infty} b_k z^k.$$

Where $b_k = a_n$ for $k \in \{0, 1 \ldots n\}$ and $b_k = 0$ for $k > n$. Now, the radius of convergence of the power series of f is all of the complex numbers.

Consider

$$g(z) = \overline{f(\bar{z})}$$

Then,

$$g(z) = \sum_{k=0}^{\infty} \overline{b_k} z^k.$$

Can you find what the radius of convergence for the power series related to g is? $\square$

For the second part, if h is differentiable at 0 then use the Cauchy-Riemann equations to show the derivative of f at zero is zero. [You can actually show a stronger statement. That f has to be constant.] To prove the converse many people where reversing there logic. Remember one key fact, the partials of the components of f must be continuous. You get this for free if f is a polynomial. In fact, this will always be true if f is differential in an open ball. But this fact wont come till later.

4.7.7 In each of the following cases, for f defined on the domain D, find the explicit formulas for $u(x,y), v(x,y)$ where $f(z) = u(x,y) + iv(x,y)$, where $z = x + iy$ and all of x, y, u, v are real.

(i) $f(z) = 1/z, D = \{z \in \mathbb{C} : z \neq 0$.

(ii) $f(z) = |z|, D = \mathbb{C}$.

(iii) $f(z) = \bar{z}, D = \mathbb{C}$.

Show that u, v satisfy the Cauchy-Riemann Equations everywhere in (i) and nowhere in $(ii), (iii)$.

4.7.9 For $z = x + iy$, let

$$f(z) = \begin{cases} \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}$$

Show that f is continuous at the origin, the Cauchy-Riemann Equations are satisfied there, yet $f'(0)$ does not exist. Why does this not contradict Theorem 4.12?

Proof. For continuity, you use the epsilon delta definition or change it to polar coordinates and let the radius approach zero. To find the partial derivatives of f use the limit definition for partial derivatives. Show $f'(0)$ doesn't exist. And lastly, this doesn't contradict theorem 4.12 since its partials are not continuous at zero. $\square$