MATH 122A HOMEWORK 2

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2.10.9 Let S be a subset of $\mathbb{C}$. If $z, w \in S$, define $z \sim w$ if and only if there is a path from z to w. Show that $\sim$ is an equivalence relation. The equivalence classes are *components* of S. If S is open and non-empty, show that each component is a domain.

Proof. The point of this question was to construct the paths needed. I am considering my paths to start from the unit interval. If you choose to start from a different interval you need to adjust the function accordingly.

$z \sim z$: The path for reflexive is the constant function $f(t) = z$

If $z \sim y$, then $y \sim z$. If $f(t)$ is the path from z to y, then consider the function $g(t) = f(1-t)$. Show the necessary conditions for this to be a path.

If $z \sim y$, and $y \sim w$ then $z \sim w$. Let f be the path from z to y and g be the path from y to w. Consider the following function:

$$h(t) = \begin{cases} f(2t) & \text{if } 0 \leq t \leq \frac{1}{2} \\ g(2t-1) & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases}.$$

Show that $h(t)$ is our desired path.

Since S is non empty there exist a component of S since we have the reflexive property. Let $[z]$ be the equivalence class of our component. Now, each component is path connected. (Why can we say this?) Since S is open, there exist an open ball around z. We showed in class that open balls are path connected. Hence, there exists some other element in our component, lets call this point y. Since S is open there exist an open ball around y which is strictly contained in S. We know that $y \sim z$. And each point in this open ball is related to y since open balls are path connected. Then each point in the open ball is related to z by transitivity. Ergo, the open ball around y is a subset of $[z]$. Since y was an arbitrary element in $[z]$ this proves that each component is open and therefore a domain. □

2.10.10 Let S be a path-connected subset of $\mathbb{C}$, and let $f : S \to \mathbb{C}$ be a continuous function. Prove that $f(S)$ is path-connected (even though it may not be open).

2.10.12 Let S be a subset of $\mathbb{C}$. A point $z \in \mathbb{C}$ is a *boundary point* of S if z is a limit point of S and also a limit point of the complement of $\mathbb{C} \setminus S$. The *boundary* ∂S of S is the set of all boundary points of S. In the following cases, describe ∂S and state whether ∂S is path-connected. Draw a picture in each case.

i $S = \{z \in \mathbb{C} : 1 < |z| < 2\}$

ii $S = \{z \in \mathbb{C} : z \neq 0\}$

iii $S = \{z \in \mathbb{C} : z = x + iy \text{ where } x, y \in \mathbb{Q}\}$

iv $S = \{z \in \mathbb{C} : 0 \leq re\, z \leq 1, 0 \leq im\, z \leq 1\}$

v $S =$ the intersection of the sets S in (iii) and (iv)

vi $S = \{z \in \mathbb{C} : z \neq iy \text{ where } y \in \mathbb{R}, y \leq 0\}$

vii $S =$ the intersection of the sets S in (vi) and (ii)

3.6.2 For what values of $z \in \mathbb{C}$ does each of the following sequences converge?

i (z^n)

Proof. We prove a small claim:

CLAIM 1. *Let* $\theta \in \mathbb{R}$*, then* $\cos(n\theta)$ *converges to 1 or -1/2.*

Proof. Assume $\cos(nt)$ converges to a. Then every subsequence also converges to a. By the double angle formula, we get:

$$\lim_{n\to\infty} \cos(2nt) = \lim_{n\to\infty} (2\cos^2(nt) - 1) \ \Rightarrow a = 2a^2 - 1$$

So our sequence $\cos(nt)$ can only converge to 1 or $-1/2$. And this is true for any real number. □

CLAIM 2. *Let* $\theta \in \mathbb{R}$*. If* $e^{ni\theta}$ *converges then* $\sin(n\theta)$ *converges to 0 and* $\cos(n\theta)$ *converges to 1.*

Proof. Use the double angle formula for sin

□

You can apply the root test and conclude that our sequence convergence in the unit circle. Another way to do this is the following. Let $z = re^{i\theta}$ for some $\theta \in \mathbb{R}$. Then $z^n = r^n e^{in\theta} = r^n \cos(n\theta) + ir^n \sin(n\theta)$. You can see that we only care about the case when $r = 1$ since the other cases hold from real analysis. Hence, we have to consider when $z = e^{2\pi it}$ where $t \in [0,1]$ We need to break this question into two cases.

[Case 1] The first case is to assume $t \in [0,1]$ and is rational. This tells use that our sequence are made up of roots of unity. Why does this sequence not converge when $t \notin \{0,1\}$?.

[Case 2] This is where I have to wave my hands at a proof here. I didn't expect an in depth proof from you all. I would have excepted a geometric heuristic argument. Assume $t \in [0,1]$ and irrational. Consider the map $f(x) = e^{2\pi ix}$ and x an irrational in the interval $[0,1]$. Then f is a continuous map and surjective onto the unit circle. So f sends dense sets to dense sets. So this is where the hand wavy part comes in [show that the lim sup and lim inf are different]. Or, we at least know that if each of these converge then $\cos(2\pi nt)$ must go to 1 and $\sin(2\pi nt)$ must go to zero. Try to convince yourself that this can't happen. But we are not in a real analysis class so its okay to move on already. □

ii (z^n/n)

Proof. You can use the ratio or root test to conclude that our sequence converges on $|z| < 1$. If you break it up into real and imaginary parts you should use the following fact, which is not to hard to believe:

LEMMA 3. *Let a_n and b_n be two real sequences. Assume a_n converges to zero and b_n is bounded, then the sequence $a_n b_n$ converges to zero.*

LEMMA 4. *Let a_n and b_n be two real sequences. Assume a_n is unbounded and b_n has a subsequence bounded away from zero, then the sequence $a_n b_n$ diverges.*

This leads to the case were we are on the unit circle. Use lemma 3 and the claim from part 1 to show $\frac{\cos(nt)}{n}$ converges for any real number t. You also need to consider $\frac{\sin(nt)}{n}$ also converges. This shows our sequence converges for any point in the closed unit disk since its real and imaginary parts converge. □

iii $(n!z^n)$

iv $(z^n/n!)$

v (z^n/n^k) where k is a positive integer.

vi $(a(a-1)\cdots(a-n+1)z^n/n!)$ where a is a fixed complex number.

Proof. When $a \in \mathbb{N}$ our sequence converges for any complex number z. For a not a natural number use the ratio test. When $|z| = 1$ it is left to you.

□

3.6.9 Show that $\sum_{n=1}^{\infty} z^{n!}$ converges for $|z| < 1$, but diverges for infinitely many z with $|z| = 1$. [We give a couple proofs for this question.]

Lets tackle the case when $|z| \neq 1$. Applying the ratio or root test confirms the series converges for $|z| < 1$. If you want to think of this as a power series centered at zero then you need to do a little more work. What are the a_n terms? Answer, they should be either a zero or a one. Now you need to use the lim sup definition. You can explicitly find the sequence. Now let us consider when $|z| = 1$.

Proof. Let $z = e^{\frac{2\pi i p}{q}}, p, q \in \mathbb{N}, (p,q) = 1$. Then $z^{n!} = 1$ whenever $n \geq q$. Therefore,

$$\sum_{n=1}^{\infty} z^n = \sum_{n=1}^{q-1} z^n + \sum_{n=q}^{\infty} z^n =$$

Clearly the tail of the series goes to infinity. So the series does not converge.

□

Proof. Let $z = e^{\frac{2\pi i}{k}}, k \in \mathbb{N}$. Then $z^{n!} = 1$ whenever $n \geq k$. Therefore, the limit does not converge to zero. By the nth term test this diverges.

□

Proof. Let $|z| = 1$, then $z = \cos(\theta) + i\sin(\theta)$ for $\theta \in [0, 2\pi]$. Then $z^{n!} = \cos(n!\theta) + i\sin(n!\theta)$ It is enough to show $\lim_{n\to} \cos(n!\theta)$ does not converge to zero.[Why is this so?]. Again, this involves more real analysis. If you are feeling nostalgic about real analysis take a crack at proof. If everything works well you can say by the nth term test, our series diverges.

□

Proof. Let $S^1 = \{z \in \mathbb{C} : |z| = 1\}$. You can show that this is a group. Any two products of this set will stay in the set by the closure property of a group. So for any $z \in S^1$ and $n \in \mathbb{N}, z^n \in S^1$. Then $z^{n!} = (z^n)^{(n-1)!} \in S^1$. Therefore, by the nth term test $\lim_{n\to\infty} z^{n!} \neq 0$.

□

Proof. We need a lemma first:

LEMMA 5. *Let z_n be a complex sequence where $z_n = x_n + iy_n$ and x_n, y_n are real sequences. Then* $\lim_{n\to\infty} z_n = 0 \iff \lim_{n\to\infty} |z_n| = 0$.

Proof. Assume ($\Leftarrow$) holds. Let $\varepsilon > 0$. Then, $\exists\, N \in \mathbb{N}$ such that for $n > N$ $||z_n| - 0| < \varepsilon$. Now $0 \leq x_n^2 + y_n^2 = |z_n| < \varepsilon^2$. Then $0 \leq |x|^2 = |x^2| < \varepsilon^2 \Rightarrow 0 \leq |x| < \varepsilon$. The argument for y_n is symmetric $\Rightarrow z_n \to 0$ as $n \to \infty$.

Assume ($\Rightarrow$) holds. Let $\varepsilon > 0$. Then $\exists\, N \in \mathbb{N}$ such that for $n > N$ $|z_n - 0| < \varepsilon$. Which is the same thing as writing $0 \leq |z_n| < \varepsilon \Rightarrow 0 - 0 \leq |z_n| - 0 < \varepsilon - 0 \Rightarrow 0 < ||z_n| - 0| < \varepsilon$.

□

Now, we are assuming z lies on the unit circle so $|z| = 1$. Then

$$lim_{n\to\infty} |z|^{n!} = lim_{n\to\infty} 1^{n!} = 1.$$

By Lemma 5 and the nth term test this series diverges for any point on the unit circle. □

3.6.10 Suppose that $\sum a_n z^n$ has radius of convergence R and let C be the circle $\{z \in \mathbb{C} : |z| = R\}$. Prove or disprove the following (which may or may not be true).

i If $\sum a_n z^n$ converges at some point on C then it converges everywhere on C.

ii If $\sum a_n z^n$ converges absolutely at some point on C then it converges everywhere on C.

iii If $\sum a_n z^n$ converges at every point on C, except possibly one, then it converges everywhere on C. (Hint: the series $\sum z^n/n$ could prove useful in this question.)

3.6.11 If $\sum a_n z^n$ has a radius of convergence R, use the formula $1/R = \limsup |a_n|^{1/n}$ to find the radius of convergence of:

i $\sum n^3 a_n z^n$

ii $\sum a_n z^{3n}$

iii $\sum a_n^3 z^n$