Math 122A Final review Hints

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1. Let $A \subset \mathbb{C}$ be an open set and $f : A \to \mathbb{C}$ an analytic function on A. Let $z_0 \in A$. Prove

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0 - \Delta z)}{2\Delta z}$$

Proof. Let $A \subset \mathbb{C}$ be an open set and $f : A \to \mathbb{C}$ an analytic function on A. Let $z_0 \in A$. Consider the limit:

$$\lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0 - \Delta z)}{2\Delta z}.$$

Since f is continuous in A and z_0 is an element of A, $f(z_0)$ exists. Adding and subtracting this term to our limit above gives the following,

$$\lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0) - (f(z_0 - \Delta z) - f(z_0))}{2\Delta z} = \lim_{\Delta z \to 0} \left[\frac{f(z_0 + \Delta z) - f(z_0)}{2\Delta z} - \frac{(f(z_0 - \Delta z) - f(z_0))}{2\Delta z} \right].$$

Applying the definition of the derivative and changing the limits to the second expression we get

$$\frac{1}{2}(f'(z_0) + (f'(z_0)).$$

This is the result as claimed.

□

2.

$$\lim_{\Delta z \to 0} \frac{\sin(\Delta z)}{\Delta z} = \lim_{\Delta z \to 0} \frac{\sin(0 + \Delta z) - \sin(0)}{\Delta z}.$$

So this question is asking the following: "what is the derivative of $\sin(z)$ at $z = 0$?". The answer is $\cos(0)$, which is 1. This is what we wanted to show.

3. Prop. Suppose f and g are analytic in a neighborhood of z_0 with zeros there of order n and k respectively. (Take the order to be 0 if the function is not 0 at z_0). Let $h(z) = f(z)/g(z)$. Then

   (a) if $k > n$, then h has a pole of order $k - n$ at z_0.

   (b) if $k = n$, then h has a removable singularity with nonzero limit at z_0.

   (c) if $k < n$, then h has a removable singularity at z_0, and setting $h(z_0) = 0$ produces an analytic function with a zero of order $n - k$ at z_0.

Proof: We know that there is neighborhood $D = \{z \in \mathbb{C} : |z - z_0| < r\}$ on which f and g factor as $f(z) = (z - z_0)^n \varphi(z)$ and $g(z) = (z - z_0)^k \psi(z)$ where φ and ψ are analytic and neither is ever 0 on D. The function $H(z) = \varphi(z)/\psi(z)$ is analytic and never equal to 0 on D(Why?). Thus, for z in the deleted neighborhood $U = D \setminus \{z_0\}$, we have

$$h(z) = \frac{f(z)}{g(z)} = \frac{(z - z_0)^n \varphi(z)}{(z - z_0)^k \psi(z)} = (z - z_0)^{n-k} H(z).$$

Our conclusions now follows by expanding our analytic function $H(z)$ around z_0.

4. Prove the Casorati-Weierstrass Theorem. Let f have an (isolated) essential singularity at z_0 and let $w \in \mathbb{C}$. Then there is a sequence $z_1, z_2, z_3, \ldots.$ in C such that $z_n \to z_0$ and $f(z_n) \to w$.

   (**Note**) This says that a deleted nbhd of an essential singularity has a dense image.

   Hint: to prove this use the same technique that an entire function has a dense image.

5. Example π: Determine the order of the pole of each of the following functions at the indicated singularity:

   (a) $(\cos z)/z^2,\ z_0 = 0$

   b) $(e^z - 1)/z^2,\ z_0 = 0$

   (c) $(z+1)/(z-1),\ z_0 = 0$

   Solution:

   (a) The function z^2 has a zero of order 2 and $\cos 0 = 1$, so $(\cos z)/z^2$ has a pole of order 2. Alternatively,

$$\frac{\cos z}{z^2} = \frac{1}{z^2}\left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \ldots\right) = \frac{1}{z^2} - \frac{1}{2!} + \frac{z^2}{4!} - \ldots,$$

   so again the pole is of order 2.

   (b) The numerator has a zero of order 1 at 0 (why?) and the denominator a zero of order 2 . The quotient thus has a simple pole. Alternatively,

$$\frac{e^z - 1}{z^2} = \frac{1}{z^2}\left[\left(1 + z + \frac{z^2}{2!} + \ldots\right) - 1\right] = \frac{1}{z} + \frac{1}{2!} + \frac{z}{3!} + \frac{z^2}{4!} + \ldots,$$

   so the pole is simple.

   (c) There is no pole since the function is analytic at 0 .

6. Prop: Let f have an isolated singularity at z_0 and let k be the smallest integer ≥ 0 such that $\lim_{z\to z_0}(z-z_0)^k f(z)$ exists. Then $f(z)$ has a pole of order k at z_0 and, if we let $\phi(z) = (z-z_0)^k f(z)$, then ϕ can be defined uniquely at $z_0 = 0$ that ϕ is analytic at z_0 and

$$\operatorname{Res}(f; z_0) = \frac{\phi^{(k-1)}(z_0)}{(k-1)!}$$

   Proof: Since $\lim_{z\to z_0}(z-z_0)^k f(z)$ exists, $\phi(z) = (z-z_0)^k f(z)$ has a removable singularity at z_0. Thus in a neighborhood of z_0,

$$\phi(z) = (z-z_0)^k f(z) = b_k + b_{k-1}(z-z_0) + \ldots + b_1 (z-z_0)^{k-1} + a_0 (z-z_0)^k +$$

   so

$$f(z) = \frac{b_k}{(z-z_0)^k} + \frac{b_{k-1}}{(z-z_0)^{k-1}} + \ldots + \frac{b_1}{(z-z_0)} + a_0 + a_1(z-z_0) + \ldots$$

   If $b_k = 0$, then $\lim_{z\to z_0}(z-z_0)^{k-1} f(z)$ exists, which contradicts the hypothesis about k. Thus, z_0 is a pole of order k. Finally, consider the expansion for $\phi(z)$, and differentiate it $k-1$ times at z_0 to obtain $\phi^{(k-1)}(z_0) = [(k-1)!]\, b_1$.