Math 122-A   Winter 2022 HOMEWORK# 6 (due February. 28)

(1) Let C be a closed, positive and simple curve. Using Green's theorem prove that $\frac{1}{2i}\int_C \bar{z}dz =$ area enclosed by C

(2) Consider the function $f(z) = (z+1)^2$ and the region R bounded by the triangle with vertices $0, 2, i$ (its boundary and interior). Find the points where $|f(z)|$ reaches its maximum and minimum value of R.

(3) Find the maximum of $|\sin(z)|$ on $[0, 2\pi] \times [0, 2\pi]$.

(4) Calculate:
(a) $\int_0^{2\pi} \frac{d\theta}{a+b\cos(\theta)} \quad 0 < b < a.$
HINT: Work backward using that $\cos(\theta) = \left(e^{i\theta} + e^{-i\theta}\right)/2$ to convert the integral into a complex integral along the curve $|z| = 1$
(b) $\int_0^{2\pi} \frac{d\theta}{(a+b\cos(\theta))^2}$
(c) $\int_0^{2\pi} \frac{\sin(\theta)d\theta}{(a+b\cos(\theta))^2}, \quad 0 < b < a.$
Hint: If you want to do a brute force method consider the following integral:

$$\operatorname{Im}\int_0^{2\pi} \frac{e^{i\theta}d\theta}{(a+b\cos(\theta))^2}.$$

This should be similar to part (b). Note that after expanding and making the substitution from the hint in a) you will be left to find the roots, assuming I didn't make a mistake, of the polynomial:

$$p(z) = bz^4 + (4a+2)z^2 + b$$

Letting $z^2 = w$ and considering $p(w)$ gives you a quadratic and you can find the roots of these no problem.

(5) Prove that if $f : \mathbb{C} \to \mathbb{C}$ is entire such that for some $n \in \mathbb{N}$

$$\lim_{|z|\to\infty} \frac{|f(z)|}{|z|^n} = M < \infty$$

then f is a polynomial of degree at most n.

(6) Let $A \subset \mathbb{C}$ be an open set and $f : A \to \mathbb{C}$ be an analytic function on A. Assuming that $z_0 \in A$ such that at z_0

$$\{z \in \mathbb{C} : |z - z_0| \le R\}, \quad R > 0$$

$f(z_0) = \frac{1}{\pi R^2}\iint_{|z-z_0|\le R} f(x+iy)dxdy.$

(7) Let $f : \mathbb{R} \to \mathbb{R}$ be defined as

$$f(x) = e^{-1/x^2} \quad \text{if} \quad x \neq 0, \quad f(0) = 0$$

Show that f is infinitely differentiable and $\forall n \in \mathbb{N} \quad f^{(n)}(0) = 0$. Verify that the power serie of f at $x = 0$ does not agree with f in any neighborhood of 0 .