Math 122-A Winter 2022 HOMEWORK# 5 (due February. 17)

1. Let $z_0 \in \mathbb{C}$ be any interior point to any positive oriented simple closed curve C. Prove
$$\oint_C \frac{dz}{z-z_0} = 2\pi i, \quad \oint_C \frac{dz}{(z-z_0)^{n+1}} = 0, n = 1, 2, 3, \ldots$$

2. Let C be the contour of the circle $|z-i| = 2$ in the positive sense. Find
   (a) $\displaystyle\oint_C \frac{dz}{z^2+4}$,
   (b) $\displaystyle\oint_C \frac{e^z dz}{z-\pi i/2}$,
   (c) $\displaystyle\oint_C \frac{\cos(z)dz}{(z^2+16)\,z}$
   (d) $\displaystyle\oint_C \frac{dz}{2z+1}$.

3. For $z \in \mathbb{C}$ with $|z| \neq 3$, denote C the contour of the circle $|z| = 3$ in the positive sense and define
$$g(z) = \oint_C \frac{2w^2 - w - 2}{w - z} dw.$$
Find the values of $g(2)$ and $g(3+2i)$.

4. Assuming that the given contour ids positive oriented, compute
   (a) $\displaystyle\oint_{|z|=3} \frac{(e^z+z)\,dz}{z-2}$,
   (b) $\displaystyle\oint_{|z|=1} \frac{e^z dz}{z^2}$,
   (c) $\displaystyle\oint_{|z|=2} \frac{dz}{z^2+z+1}$,
   (d) $\displaystyle\oint_{|z|=1} \frac{dz}{z^2-1}$.

   DEFINITION: A $f : \mathbb{C} \to \mathbb{C}$ is an ENTIRE function if f is analytic in all $\mathbb{C}$.

5. Prove that if f is entire and there exist $z_0 \in \mathbb{C}$ and $r > 0$ such that
$$f(\mathbb{C}) \cap \{z \in \mathbb{C} : |z - z_0| < r\} = \emptyset$$
then f is a constant function.

6. Identify all entire functions f such that $\forall z \in \mathbb{C} \quad |f(z)| \leq 2|z|$.

Proof. I will prove this using Morera's theorem. Consider the function

$$g(z) = \begin{cases} \dfrac{f(z)}{z} & z \neq 0 \\ f'(0) & z = 0 \end{cases}$$

Clearly, our function g is bounded by $\max\{2, f'(0)\}$ and is analytic on the whole complex plane **except for the point** $z = 0$.

CLAIM 1. *g is continuous at $z = 0$.*

Proof. You should be able to provide a proof. Remember what we are using about f. It is bounded by what and how can you use that fact near zero?

□

Let γ be a simple path around zero. Let $\epsilon > 0$ and a small enough so that the circle γ_1 of radius ϵ and center 0 lies inside of γ. By the Deformation Theorem

$$\int_\gamma g = \int_{\gamma_1} g$$

By assumption, g is bounded by 2 near 0. Hence, for ϵ small enough

$$\left| \int_{\gamma_1} g \right| \leq 2\pi\epsilon 2.$$

This hold for all small enough ϵ. Ergo,

$$\int_\gamma g = 0.$$

Now g satisfies the requirements for Morera's Theorem. This now gives us that g is analytic on the complex plane, hence entire. We also know that g is bounded; invoking Liouville's Theorem we have that g must be constant. It must be the case that $g = az$ for some $a \in \mathbb{C}$. Given our assumption $|a| \leq 2$. This shows the result.

□