Math 34B Midterm Practice

1. Answer the following

(a) Find a function $h(t)$ such that $h'(t) = 8t - 2$ and $h(0) = 1$.

$$\int h'(t)\,dt = \int 8t-2\,dt = 4t^2 - 2t + C$$

$$h(0) = 1 \longrightarrow 4\cdot 0^2 - 2\cdot 0 + C = 1 \rightarrow C = 1 \rightarrow \boxed{h(t) = 4t^2 - 2t + 1}$$

(b) What is the average value $f(x) = \dfrac{1}{\sqrt{x}}$ on the interval from $x = 1$ to $x = 4$?

$$\frac{1}{4-1}\int_1^4 \frac{1}{\sqrt{x}}\,dx = \frac{1}{3}\int_1^4 x^{-\frac{1}{2}}\,dx = \frac{1}{3}\left[2x^{\frac{1}{2}}\right]_1^4$$

$$= \frac{1}{3}(2\cdot 2 - 2\cdot 1) = \boxed{\frac{2}{3}}$$

(c) Compute $\dfrac{d}{dx}(x^3\cos(x))$.

Product rule: $\frac{d}{dx}(x^3\cos(x)) = \boxed{3x^2\cos(x) - x^3\sin(x)}$

(d) For what value of n is $\displaystyle\int_0^1 x^n\,dx = \frac{1}{3}$?

$$\int_0^1 x^n\,dx = \left[\frac{x^{n+1}}{n+1}\right]_0^1 = \frac{1^{n+1}}{n+1} = \frac{1}{n+1} = \frac{1}{3} \rightarrow \boxed{n=2}$$

2. The amount $m(t)$ of carbon-14 in a body decays according to the differential equation

$$m'(t) = -0.000121m(t)$$

Write down an exponential function for $m(t)$. Find the half-life of carbon-14 using this information.

$m(t) = Ae^{kt}$

$\rightarrow m'(t) = Ake^{kt}$

Plugging in, $\cancel{A}ke^{kt} = -.000121\cancel{A}e^{kt} \rightarrow k = -.000121$

$\rightarrow \boxed{m(t) = Ae^{-.000121t}}$ (A can be any #)

(you don't have to do this whole process, if you see the right $m(t)$ right away)

Half-life: Since $m(0) = A$, we solve for t:

$\frac{1}{2}\cancel{A} = \cancel{A}e^{-.000121t} \rightarrow \ln(\frac{1}{2}) = -.000121t \rightarrow \boxed{t = \ln(\frac{1}{2}) \cdot \frac{1}{-.000121}}$

In 1989, some human skeletal remains of several bodies were found at a construction site in San Francisco, California. After testing the bones, it was determined that these skeletons had only 88% of the carbon-14 expected in a living person. Use the above model to determine what year these people died.

$0.88\cancel{A} = \cancel{A}e^{-.000121t} \rightarrow \ln(0.88) = -.000121t$

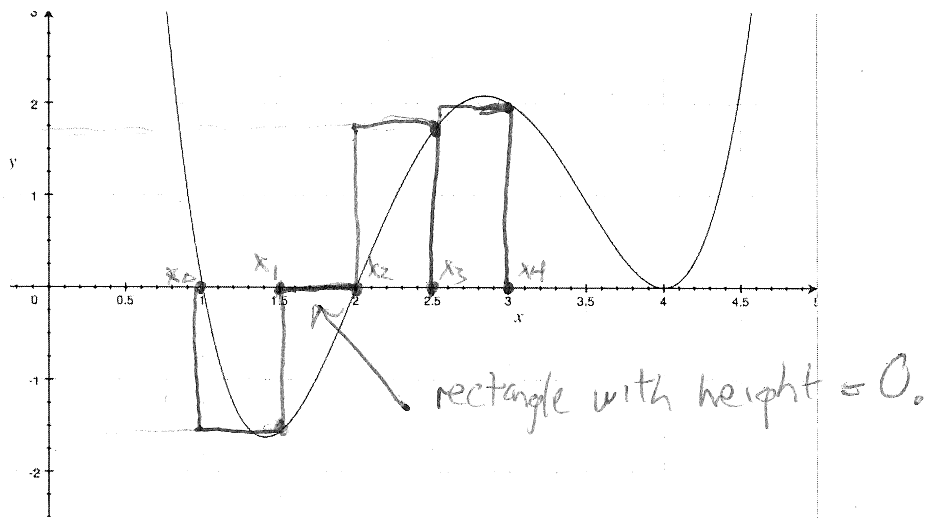
$\rightarrow t = \frac{\ln(0.88)}{-.000121}$.

This value of t tells the length of time it takes for the amount of carbon 14 to decay to 88% of what is in a living person so the people died in

$1989 - \frac{\ln(0.88)}{-.000121} = 932.52$, so they died during the year 932.

3. Mark on the graph the 4 rectangles you would use for a right Riemann sum to approximate:

$$\int_1^3 f(x)\,dx \approx \sum_{n=1}^{4} f(x_n)\Delta x$$



a. What is Δx? What are x_0, x_1, x_2, x_3 and x_4?

Δx = width of each rectangle $= 0.5 = \frac{3-1}{4}$

$x_0 = 1,\ x_1 = 1.5,\ x_2 = 2,\ x_3 = 2.5,\ x_4 = 3$

b. Write out the 4 terms of the Riemann sum and then compute the sum.

$0.5(-1.5) + 0.5(0) + 0.5(2.75) + 0.5(2)$

$= -0.75 + 0 + 1.375 + 1 = 1.625$ (you may prefer fractions by hand)

4. Use a left Riemann sum with 3 rectangles to approximate

$$\int_1^5 \frac{1}{x}\,dx$$

$\Delta x = \frac{5-1}{3} = \frac{4}{3}$

left endpoint $= x_i = a + (i-1)\Delta x$

$= 1 + (i-1)\frac{4}{3}$

$= \frac{4}{3}i - \frac{1}{3}$

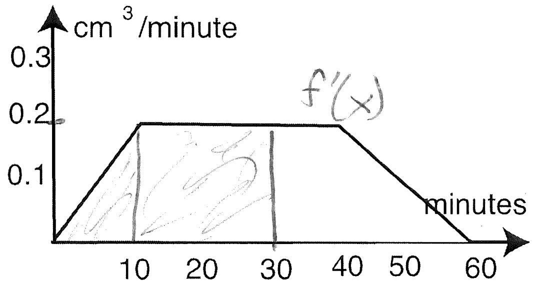
$\sum_{n=1}^{3} \Delta x f(x_i) = \frac{4}{3}\left(\frac{1}{\frac{4}{3}-\frac{1}{3}}\right) + \frac{4}{3}\left(\frac{1}{\frac{8}{3}-\frac{1}{3}}\right) + \frac{4}{3}\left(\frac{1}{\frac{12}{3}-\frac{1}{3}}\right)$

$= \frac{4}{3} + \frac{4}{3}\left(\frac{3}{7}\right) + \frac{4}{3}\left(\frac{3}{11}\right)$

$= \boxed{\frac{4}{3} + \frac{4}{7} + \frac{4}{11}\left(= \frac{524}{231}\right)}$

or use $\sum_{n=0}^{2} \Delta x f(x_i)$ with $x_i = a + i\Delta x = 1 + \frac{4}{3}i$

5. A hospital patient is on a drip. The graph shows the rate that saline solution enters the patient.



(i) How many cm^3 of saline entered in the first 30 minutes ?

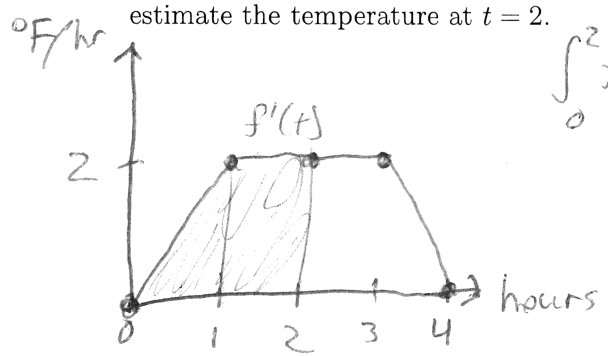
$$= \frac{1}{2}10(.2) + 20(.2) = 1 + 4 = \boxed{5 \text{ cm}^3}$$

(ii) What was the average rate in cm^3 per hour that saline entered the patient during the first half hour?

$$= \frac{1}{30-0}\int_0^{30} f'(x)\,dx = \frac{1}{30}\cdot 5 = \boxed{\frac{1}{6}\ \text{cm}^3/\text{hr}}$$

t	0	1	2	3	4
°F/hr	0	2	2	2	0

6. The table shows the rate that temperature is increasing between $t = 0$ and $t = 4$ measured in hours. The initial temperature is 50° F. Sketch a piecewise linear (made up of straight line segments) graph of the rate of increase of temperature. Use this to estimate the temperature at $t = 2$.



$$\int_0^2 f'(t)\,dt = f(2) - f(0)$$

$$= (\text{temp. at } t=2) - 50°F = \frac{1}{2}1\cdot 2 + 1\cdot 2$$

$$= 3$$

$\Rightarrow$ temp at $t=2$ is 53°F

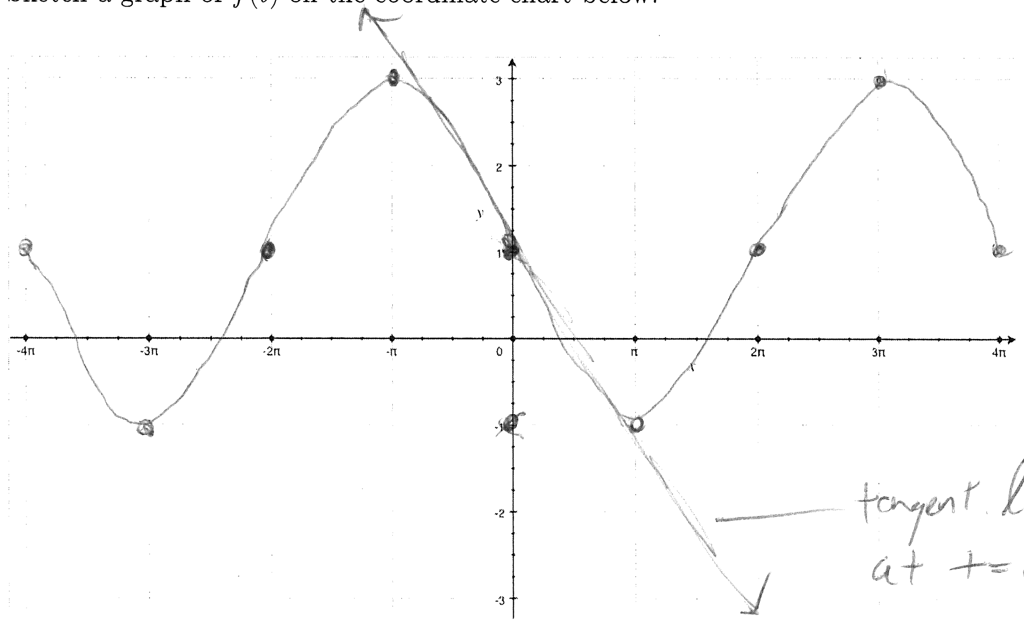
shift up 1 A K shift left 2π units.

7. Let $f(t) = 1 + 2\sin(\frac{1}{2}(t + 2\pi))$. (What's the period, amplitude, and frequency?)

(a) Sketch a graph of $f(t)$ on the coordinate chart below.

Period $= \frac{2\pi}{K} = \frac{2\pi}{\frac{1}{2}} = 4\pi$

Frequency $= \frac{1}{4\pi}$



tangent line at $t=0$

(b) Sketch the tangent line to your graph at $t = 0$, then find the equation for this tangent line.

$f(t) = 1 + 2\sin(\frac{1}{2}(t + 2\pi))$

Chain rule

$f'(t) =$ slope of tangent line $= 2\cos(\frac{1}{2}(t+2\pi))\frac{d}{dt}(\frac{1}{2}(t+2\pi))$

$= 2\cos(\frac{1}{2}(t+2\pi))(\frac{1}{2})$

$= \cos(\frac{1}{2}(t+2\pi))$

So $f'(0) = \cos(\pi) = -1$

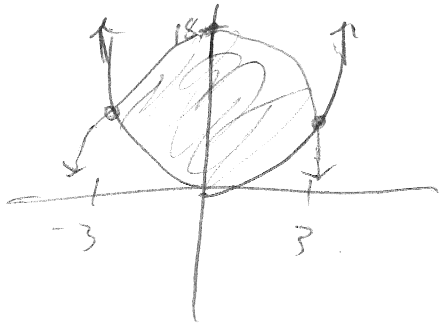
Point on tangent line is $(0, f(0)) = (0, 1+2\sin(\pi))$

$= (0,1)$ ← use your graph if accurate

So equation of tangent line is $=$ y-intercept.

$y = -x + 1$

8. Find the area between the curves $y = x^2$ and $y = 18 - x^2$.



Intersect when $x^2 = 18 - x^2 \to x^2 = 9 \to x = \pm 3$

$$\int_{0}^{3} (18 - x^2) - x^2 \, dx = \int_{-3}^{3} 18 - 2x^2 \, dx$$

$$= \left[18x - \frac{2}{3}x^3 \right]_0^3 = 54 - 18 - (0 - 0)$$

$$= 36$$

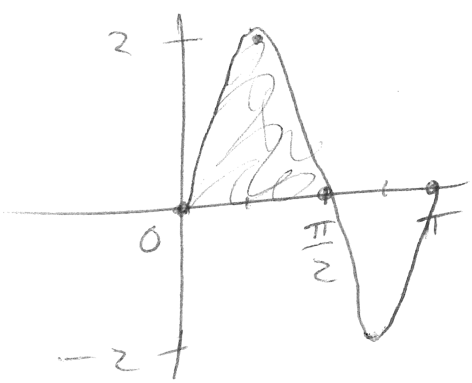
(or just integrate from -3 to 3)

so area $= 36 \cdot 2 = \boxed{72}$ units^2 by symmetry

9. Find the (positive) area under one arc of $2\sin(2x)$. First find the period.

$$\text{Period} = \frac{2\pi}{k} = \frac{2\pi}{2} = \pi$$

sketch:



$$\int_0^{\frac{\pi}{2}} 2\sin(2x) = \left[-\frac{2}{2}\cos(2x) \right]_0^{\frac{\pi}{2}}$$

$$= -\cos(\pi) - (-\cos(0))$$

$$= --1 + 1$$

$$= 1 + 1 = \boxed{2}$$

10. Let $f(x) = \frac{1}{3}x^3 + x^2 - 15x$.

(a) Find all the critical points of $f(x)$.

$ax^2+bx+c=0$

use quad. formula $\frac{-b \pm \sqrt{b^2-4ac}}{2a}$

if you can't factor this

$f'(x) = x^2 + 2x - 15 = 0$

$(x+5)(x-3) = 0$

$\rightarrow$ critical pts are $x = -5$, $x = 3$.

(b) Classify each critical point as a maximum or a minimum using the second derivative test (write "inconclusive" if the second derivative test is inconclusive).

$f''(x) = 2x + 2$

$f''(-5) = -8 < 0 \rightarrow$ local max at $x = -5$

$f''(3) = 8 > 0$ local min at $x = 3$

(c) Sketch a rough graph of what the graph of $f(x)$ *might* look like based on your answer for (b). Be sure to label your axes.