Midterm 2 (Yellow)

1(A) $f(x) = \dfrac{\sqrt[3]{x} + x}{x^2} = \dfrac{x^{\frac{1}{3}}}{x^2} + \dfrac{x}{x^2}$

$= x^{\frac{1}{3} - \frac{6}{3}} + x^{-1}$

$= x^{\frac{-5}{3}} + x^{-1}$

so $f'(x) = \dfrac{-5}{3} x^{\frac{-8}{3}} - x^{-2}$

1(B). $g(x) = 3^x$

$g'(x) = 3^x \ln(3)$ using formula

or $g(x) = 3^x = e^{\ln(3^x)} = e^{x \ln(3)}$

$g'(x) = \ln(3) e^{x \ln(3)}$

$= 3^x \ln(3)$, if you forget.

1(C). $h(x) = (\ln(x))^{100}$

$h'(x) = 100(\ln(x))^{99} \cdot \dfrac{1}{x}$.

1(D). $F(x) = \dfrac{5x}{x^2-1}$

$$F'(x) = \boxed{\dfrac{(x^2-1)(5) - 5x(2x)}{(x^2-1)^2} = \dfrac{-5x^2-5}{(x^2-1)^2}}$$

1(E). $G(x) = \dfrac{3x^2\tan(x)}{\sec(x)}$

$$= \dfrac{3x^2 \frac{\sin x}{\cos x}}{\frac{1}{\cos x}} = 3x^2 \frac{\sin x}{\cancel{\cos x}} \cdot \cancel{\cos x}$$

So $G'(x) = \boxed{3x^2\cos(x) + 6x\sin(x)}$

2. $x^3 + y^3 = 6xy \quad (3,3)$

$$3x^2 + 3y^2 \cdot \frac{dy}{dx} = 6x\frac{dy}{dx} + 6y$$

$$\rightarrow 3y^2 \frac{dy}{dx} - 6x\frac{dy}{dx} = 6y - 3x^2$$

$$\rightarrow \frac{dy}{dx}(3y^2 - 6x) = 6y - 3x^2$$

$$\rightarrow \frac{dy}{dx} = \frac{6y - 3x^2}{3y^2 - 6x}$$

so slope of tangent line at $(3,3)$ is $\dfrac{6(3) - 3(3^2)}{3(3^2) - 6(3)}$

$$= -1$$

We know $(3,3)$ is on the tangent line,

So the equation of the tangent line is

$$y - 3 = -(x-3)$$

or

$$y = -x + 6$$

3. If $f(x) = x^5 + x^3 + x$ and $g(x) = f^{-1}(x)$, find $g'(3)$.

First note that $f(1) = 3$.

We know

$$g(f(x)) = x \quad \left(\begin{array}{c}\text{cancellation equations}\\ \text{for inverses}\end{array}\right)$$

$$\rightarrow g'(f(x)) \cdot f'(x) = 1 \quad (\text{chain rule})$$

Since $f'(x) = 5x^4 + 3x^2 + 1$, $f'(1) = 9$, we have

$$g'(f(1)) \cdot f'(1) = 1$$

$$\rightarrow g'(3) \cdot 9 = 1$$

$$\rightarrow \boxed{g'(3) = \frac{1}{9}}$$

4. $f(x) = (\cos(x))^{\frac{1}{x}}$

$$y = (\cos(x))^{\frac{1}{x}}$$

$$\ln y = \frac{1}{x}\ln(\cos x)$$

$$\frac{1}{y}\cdot y' = \frac{1}{x}\left(\frac{1}{\cos x}\cdot -\sin x\right) + \frac{-1}{x^2}(\ln(\cos x))$$

$$\frac{1}{y}\cdot y' = \frac{-\tan x}{x} - \frac{\ln(\cos x)}{x^2}$$

$$y' = y\left[\frac{-\tan x}{x} - \frac{\ln(\cos x)}{x^2}\right]$$

$$\boxed{f'(x) = (\cos x)^{\frac{1}{x}}\left[\frac{-\tan x}{x} - \frac{\ln(\cos x)}{x^2}\right]}$$

(alternatively, start by writing

$$f(x) = (\cos x)^{\frac{1}{x}} = e^{\ln[(\cos x)^{\frac{1}{x}}]}$$

$$= e^{\frac{1}{x}\ln(\cos x)}$$

then find $f'(x)$)

5. Given: $2x+y=-4$ $\Big\}$ tangent line equation to $y=g(x)$ at $x=8$

$\rightarrow$ $y = -2x - 4$

$\rightarrow$ $g'(8) = -2.$

Want: Slope of tangent line to $y=g(x^3)$ at $x=2$,

Which is

$[g(x^3)]'$, at $x=2$.

Using chain rule,

$[g(x^3)]' = g'(x^3) \cdot 3x^2$ at $x=2$

$= g'(8) \cdot 12$

$= -2 \cdot 12 = -24$

6. Given: radius increasing at 2 ft/sec.

$$\frac{dr}{dt} = 2 \text{ ft/sec.}$$

Want: Rate of change of circumference when diameter is 16 feet.

$$\left(\frac{dC}{dt}\right).$$

Use $C = 2\pi r$

$$\rightarrow \frac{dC}{dt} = 2\pi \cdot \frac{dr}{dt}$$

$$= 2\pi\left(2 \text{ ft/sec}\right)$$

$$= \boxed{4\pi \text{ ft/sec} \quad \text{(when diameter is 16 feet)}}$$