MATH 3B Final practice problems

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1. A honeybee population starts with 100 bees and increases at a rate of $n'(t)$ bees per week. What does $100 + \int_0^4 v'(t)\ \mathrm{d}t$ represent?

   Solution: By FTC2/Net Change theorem,

   $$100 + \int_0^4 v'(t)dt = 100 + n(4) - n(0) = 100 + n(4) - 100 = n(4),$$

   and $n(4)$ is just the bee population after 4 weeks.

2. Compute the following:

   (a) $\int e^{-x} \cos(2x)\ \mathrm{d}x$

   Solution: Apply integration by parts twice and get $\int e^{-x} \cos(2x)\ \mathrm{d}x$ by itself.

   (b) $\int \frac{1+x^2}{\sqrt{3x+x^3}}\ \mathrm{d}x$

   Solution: Let $u = 3x + x^3$. Then $\frac{1}{3}du = (1+x^2)dx$ and the rest is straightforward.

   (c) $\int \cos(\sqrt{x})\ \mathrm{d}x$

   Solution: Let $u = \sqrt{x}$. Then $du = \frac{1}{2\sqrt{x}}dx \Rightarrow 2\sqrt{x}du = 2udu = dx$. Then integration by parts.

   (d) $\frac{d}{dx}\int_x^{x^2} t^3\ \mathrm{d}t$

   Solution: By FTC1 and the chain rule,

   $$\frac{d}{dx}\int_x^{x^2} t^3 dt = \frac{d}{dx}\left[\int_0^{x^2} t^3 dt - \int_0^{x} t^3 dt\right] = (x^2)^3 2x - x^3 = 2x^7 - x^3.$$

   (e) $\int \frac{(ln(x))^3}{x}\ \mathrm{d}x$

   Solution: Let $u = ln(x)$, so $du = \frac{1}{x}dx$...

(f) $\displaystyle\int \frac{x^3 - \sqrt{x}}{x^2}\ \mathrm{d}x$

Solution: Write as two fractions and integrate: $\frac{x^3-\sqrt{x}}{x^2} = x - x^{-\frac{3}{2}}$.

(g) $\displaystyle\int \sin^3(x)\cos^2(x)\ \mathrm{d}x$

Solution: $\displaystyle\int \sin^3(x)\cos^2(x)\ \mathrm{d}x = \int (1- \cos^2(x))\sin(x)\cos^2(x)\ \mathrm{d}x$.
Let $u = \cos(x) \Rightarrow -du = \sin(x)dx$...

(h) $\displaystyle\int_0^1 \frac{4}{4x-1}\ \mathrm{d}x$

Solution: Notice that the integrand has a discontinuity at $x = 1/4$, so

$$\int_0^1 \frac{4}{4x-1} dx = \lim_{t\to 1/4^-} \int_0^t \frac{4}{4x-1} dx + \lim_{t\to 1/4^+} \int_t^1 \frac{4}{4x-1} dx.$$

If you show that one of these integrals diverge, then the whole thing does (both diverge).

(i) $\displaystyle\int \frac{1}{x^2-1} dx$

Solution: Factor and do partial fractions...

3. Consider the region $R = \{(x, y) : x \geq 1, 0 \leq y \leq 1/x\}$.

(a) Show that the region R has infinite area.

Solution: The area of the region R is

$$\int_1^\infty 1/x dx = \lim_{t\to\infty} \int_1^t 1/x dx = \lim_{t\to\infty} [ln(x)]_1^t = \lim_{t\to\infty} ln(t) - 0 = \infty.$$

(b) Find the volume of the solid obtained by rotating R about the x-axis. It is finite, which is kinda crazy given part (a), huh?

Solution: Method 1: Disks

$$\pi \int_1^\infty r^2 dx = \pi \lim_{t\to\infty} \int_1^t \frac{1}{x^2} dx = \pi \lim_{t\to\infty} \left[-\frac{1}{x}\right]_1^t = \pi(0 - (-1)) = \pi.$$

Solution: Method 2: Shells

$$2\pi \int_0^1 rh dy = 2\pi \int_0^1 y\left(\frac{1}{y} - 1\right) dy = 2\pi \int_0^1 1 - y dy = 2\pi \left[y - \frac{y^2}{2}\right]_0^1 = \pi.$$

Notice that for the first method we had an improper integral, and for the second we didn't, but we did need to solve for x in terms of y.

4. Find the work required to lift a chain lying on the ground to a height of 10 feet if the chain weighs 20 pounds and is 10 feet long.

Solution: Method 1: Chop up the object. We chop up the chain into pieces of length Δx ft. Each piece weighs Δx ft $\times$ 2 lb/ft $= 2\Delta x$ lbs. A piece x ft above ground needs to be lifted up another $10 - x$ ft, so the work required for each piece of chain is $2(10 - x)\Delta x$. Adding up the work required for each piece gives $\int_0^{10} 2(10 - x)dx = 100J$.

Solution: Method 2: Chop up the distance. Let x be the height above the ground. We'll add up the work required to lift the chain each tiny distance Δx. If we're at a height of x feet, we have to lift x feet of chain and the rest is coiled up on the ground. The force to move each tiny distance $= x$ feet $\times$ 2 lb/ft $= 2x$ lb, so the total work is $W = \int_0^{10} 2x \text{ d}x = 100J$.

5. Evaluate $\displaystyle\int_0^\infty \frac{1}{x-2} \text{ d}x$ or show that the integral diverges.

Solution: Notice we are integrating over an infinite interval and $f(x) = \frac{x}{x-2}$ is discontinous at $x = 2$, which is in our interval of integration. So we break up the integral as

$$\int_0^\infty \frac{1}{x-2}dx = \lim_{t\to 2^-} \int_0^t \frac{1}{x-2}dx + \lim_{t\to 2^+} \int_t^3 \frac{1}{x-2}dx + \lim_{t\to\infty} \int_3^t \frac{1}{x-2}dx.$$

If one of the integrals on the right diverges, then $\displaystyle\int_0^\infty \frac{1}{x-2}dx$ diverges. The last one looks a lot like $\int \frac{1}{x}dx$ so we show it diverges:

$$\lim_{t\to\infty} \int_3^t \frac{1}{x-2}dx = \lim_{t\to\infty} [ln|x-2|]_3^t = \lim_{t\to\infty} ln|t-3| - 0 = \infty.$$

Thus $\displaystyle\int_0^\infty \frac{1}{x-2}dx$ diverges.

6. Find the volume of the solid obtained by rotating the region bounded by the curves $y = 2x$ and $y = x^2$ about the x-axis.

Solution: The curves intersect at $x = 0$. We find the other intersection point: $2x = x^2 \Rightarrow x = 2$. Graph them or note that $2x \geq x^2$ on $[0, 2]$. Washers seem the best choice as then we'll integrate with respect to x and we already have y in terms of x:

$$\pi \int_0^2 r_o^2 - r_i^2 dx = \pi \int_0^2 (2x)^2 - (x^2)^2 dx = \pi \int_0^2 4x^2 - x^4 dx = \pi \left[\frac{4x^3}{3} - \frac{x^5}{5}\right]_0^2 = \pi \left(\frac{32}{3} - \frac{32}{5}\right).$$

7. Let $f(x) = x^2$. Approximate $\displaystyle\int_0^4 x^2 \text{ d}x$ using a right hand Riemann sum and 4 subintervals.

Solution: We approximate the area by adding up the area of 4 rectangles, whose width is $\Delta x = \frac{b-a}{n} = \frac{4-0}{4} = 1$, and whose height is determined by the value of $f(x)$ at the right endpoints of each rectangle, $x = 1, 2, 3$, and 4. This yields

$$\int_0^4 x^2 dx \approx \Delta x[f(1) + f(2) + f(3) + f(4)] = (1)[1^2 + 2^2 + 3^2 + 4^2] = 30.$$