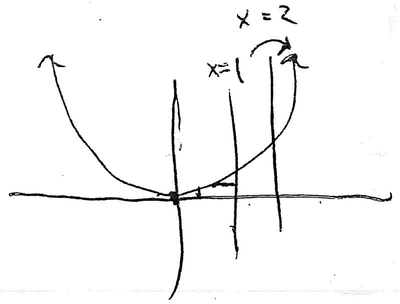
A. 1.



$y = x^4 \rightarrow x = y^{\frac{1}{4}}$

Washers: $\cancel{\text{Disks}}$ $\pi(r_{outer}^2 - r_{inner}^2)$

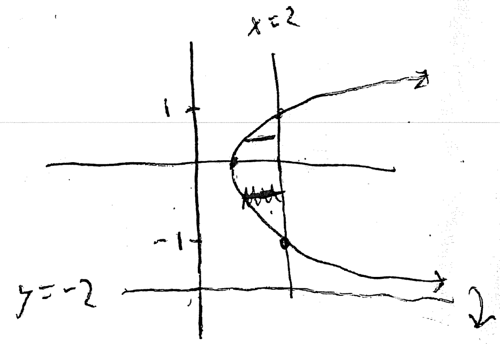
$= \pi((2 - y^{\frac{1}{4}})^2 - (2-1)^2)$

$\pi \int_0^1 (2 - y^{\frac{1}{4}})^2 - 1 \, dy = \frac{7\pi}{15}$.

Shells: $2\pi r h = 2\pi(2-x)x^4$

$2\pi \int_0^1 (2-x)x^4 \, dx = \frac{7\pi}{15}$

2.



$x = y^2 + 1$, $x = 2$ about $y = -2$

$y^2 + 1 = 2$

$y^2 - 1 = 0$

$(y+1)(y-1) = 0$

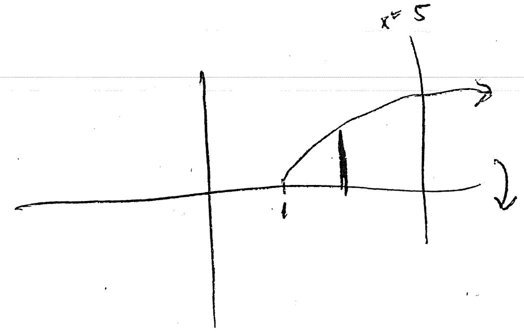
$y = \pm 1$

~~Shells: $2\pi r h = 2\pi$~~

Shells: $2\pi r h = 2\pi(2+y)(2 - (y^2+1)) = 2\pi(2+y)(1-y^2)$

$2\pi \int_{-1}^{1} (2+y)(1-y^2) \, dy = \frac{16\pi}{3}$.

3.

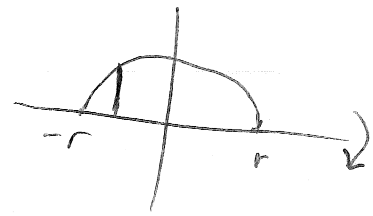


Disks: $\pi r^2 = \pi \sqrt{x-1}^2 = \pi(x-1)$

$\pi \int_1^5 x - 1 \, dx = 8\pi$.

4. $x^2+y^2=r^2$. Rotate $y=\sqrt{r^2-x^2}$ about x-axis, $-r \le x \le r$.

Disks: $\pi r^2 = \pi \sqrt{r^2-x^2}^2 = \pi(r^2-x^2)$



$$2 \cdot \pi \int_0^r r^2-x^2 \, dx = \dots = \frac{4}{3}\pi r^3.$$

Shells: $2\pi r h = 2\pi y(2\sqrt{r^2-y^2})$

$$4\pi \int_0^r y\sqrt{r^2-y^2}\, dy = \frac{-1}{2}\cdot 4\pi \int_{r^2}^{0} u^{\frac{1}{2}}\, du = \dots = \frac{4}{3}\pi r^3.$$

$u=r^2-y^2$

$\frac{-1}{2}\, du = y\, dy$

B. 1. $\frac{dx}{dy} = (y-1)^{\frac{1}{2}} \to ds = \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy = \sqrt{1+y-1}\, dy = \sqrt{y}\, dy$

$$L = \int ds = \int_1^4 y^{\frac{1}{2}}\, dy = \frac{14}{3}.$$

2. $\frac{dx}{dt} = 1-\cos t$, $\frac{dy}{dt} = \sin t$

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\, dt = \sqrt{1-2\cos t+\cos^2 t+1+2\sin t+\sin^2 t}\, dt$$

$$= \sqrt{2(\sin t-\cos t)}\, dt$$

$$= \sqrt{1-2\cos t+\cos^2 t+\sin^2 t}\, dt = \sqrt{2-2\cos t}\, dt = \sqrt{4\sin^2\left(\frac{t}{2}\right)}\, dt$$

$$= 2\sin\left(\frac{t}{2}\right) dt$$

$$L = \int ds = \int_0^{2\pi} 2\sin\left(\frac{t}{2}\right) dt = \dots = 8.$$

(use $1-\cos t = 2\sin^2\frac{t}{2}$)

C. 1. $S = 2\pi \int y \, ds$. $ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt$.

$\frac{dx}{dt} = -3\cos^2 t \sin t$, $\frac{dy}{dt} = 3\sin^2 t \cos t$

$\rightarrow ds = \sqrt{9\cos^4 t \sin^2 t + 9\sin^4 t \cos^2 t} \, dt = 3\sqrt{\cos^2 \sin^2 t(\cos^2 t + \sin^2 t)} \, dt$

$= 3\cos t \sin t \quad dt$

$S = 2\pi \int_0^{\frac{\pi}{2}} \sin^3 t (3\cos t \sin t) \, dt = \ldots = \frac{6\pi}{5}$ (let $u = \sin t$)

2. $S = 2\pi \int y \, ds$. $ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx$

$\frac{dy}{dx} = 3x^2 \longrightarrow ds = \sqrt{1 + 9x^4} \, dx$

$S = 2\pi \int_0^2 x^3 \sqrt{1 + 9x^4} \, dx = \ldots = 64.63\pi$ (let $u = 1 + 9x^4$)

3. $ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \, dy = \sqrt{1 + y^2(y^2+2)} \, dy = \sqrt{y^4 + 2y^2 + 1} \, dy$

$\frac{dx}{dt} = \frac{1}{2}(y^2+2)^{\frac{1}{2}} \cdot 2y = y(y^2+2)^{\frac{1}{2}}$ $\quad = \sqrt{(y^2+1)^2} \, dy$.

$= y^2 + 1 \, dy$

$S = 2\pi \int y \, ds = 2\pi \int_1^2 y(y^2+1) \, dy = \ldots = \frac{21\pi}{2}$.

4. Rotate $y = \sqrt{r^2 - x^2}$ about x-axis, $-r \leq x \leq r$.

$\frac{dy}{dx} = \frac{-\cancel{2}x}{\cancel{2}\sqrt{r^2 - x^2}} \rightarrow ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx = \sqrt{1 + \frac{x^2}{r^2 - x^2}} \, dx = \sqrt{\frac{r^2}{r^2 - x^2}} \, dx = \frac{r}{\sqrt{r^2 - x^2}} \, dx$.

$S = 2\pi \int y \, ds = 2\pi \int_{-r}^{r} \cancel{\sqrt{r^2 - x^2}} \cdot \frac{r}{\cancel{\sqrt{r^2 - x^2}}} \, dx = \ldots = 4\pi r^2$.

D. 1. Use $\frac{1}{2}\int_0^{2\pi} x(t)y'(t) - y(t)x'(t)\, dt$. $\quad x'(t) = -3\sin t$, $\quad y'(t) = 5\cos t$

$$= \frac{1}{2}\int_0^{2\pi} 15\cos^2(t) + 15\sin^2 t\, dt = \frac{15}{2}\int_0^{2\pi} dt = 15\pi.$$

E 1. $\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \frac{k}{\sqrt{n^2+kn}} = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \frac{k}{\sqrt{n^2+kn}} \cdot \frac{\frac{1}{n}}{\frac{1}{\sqrt{n^2}}} = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \frac{\frac{k}{n}}{\sqrt{1+\frac{k}{n}}}$

$$= \int_0^1 \frac{x}{\sqrt{1+x}}\, dx$$

$u = 1+x \to x = u-1$

$du = dx$

$$= \int_1^2 \frac{u-1}{u^{\frac{1}{2}}}\, du = \int_1^2 u^{\frac{1}{2}} - u^{-\frac{1}{2}}\, du$$

$$= \left[\frac{2}{3}u^{\frac{3}{2}} - 2u^{\frac{1}{2}}\right]_1^2 = \frac{2}{3}\cdot 2^{\frac{3}{2}} - 2\cdot 2^{\frac{1}{2}} - \frac{2}{3} + 2.$$

2. Take $\ln$ once then e at the end.

$$\lim_{n\to\infty} \ln\left[\left(\frac{1}{n}\right)^1\left(\frac{2}{n}\right)^2\cdots\left(\frac{n}{n}\right)^n\right]^{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{1}{n^2}\left[\ln\frac{1}{n} + 2\ln\frac{2}{n} + \dots + n\ln\frac{n}{n}\right]$$

$$= \lim_{n\to\infty} \frac{1}{n^2}\sum_{k=1}^{n} k\ln\frac{k}{n} = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \frac{k}{n}\ln\frac{k}{n}$$

$u = \ln x \quad v = \frac{x^2}{2}$

$du = \frac{1}{x}dx \quad dv = x$

$$= \int_0^1 x\ln x = \frac{x^2}{2}\ln x - \frac{1}{2}\int x\, dx$$

$$= \lim_{t\to 0^+} \left[\frac{x^2}{2}\ln x - \frac{1}{4}x^2\right]_t^1$$

$$= \lim_{t\to 0^+} \frac{-1}{4} - \frac{t^2}{2}\ln t + \frac{1}{4}t^2$$

$$= \frac{-1}{4} - \frac{1}{2}\lim_{t\to 0^+} t^2\ln t = \dots$$

$$\ldots = \frac{-1}{4} - \frac{1}{2}\lim_{t\to 0^+} \frac{\ln t}{\frac{1}{t^2}} \overset{\text{l'Hopital}}{=} \frac{-1}{4} - \frac{1}{2}\lim_{t\to 0^+} \frac{1}{t}\cdot\frac{t^{3}}{-2}$$

$$= \frac{-1}{4}, \text{ so limit is } e^{-\frac{1}{4}}.$$

3. $\lim_{n\to\infty} \frac{1}{n\sqrt{n}} \sum_{k=1}^{n} \frac{k}{\sqrt{n+k}} = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \frac{k}{\sqrt{n^2+kn}}$. This is the same as #1 now, ~~without the typo~~.

4. $\lim_{n\to\infty} \sum_{k=1}^{n} \sqrt{\frac{k}{n^3}+\frac{2}{n^2}} = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \sqrt{n^2}\sqrt{\frac{k}{n^3}+\frac{2}{n^2}}$

$$= \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n}\sqrt{\frac{k}{n}+2}$$

$$= \int_0^1 \sqrt{x+2}\,dx = \ldots = 2\sqrt{3} - \frac{4\sqrt{2}}{3} \approx 1.58$$

$(u = x+2)$

5. $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{k}{n^2} e^{\frac{k^2}{n^2}} = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \frac{k}{n} e^{\left(\frac{k}{n}\right)^2} = \int_0^1 x e^{x^2}\,dx$

$$= \ldots = \frac{1}{2}(e-1) \quad (u = x^2)$$

F. For ① - ③, see improper integrals worksheet solutions.

4. For $0 < t \le 1$, $\sec^2 x > 1 \rightarrow \frac{\sec^2 x}{t^{\frac{3}{2}}} > \frac{1}{t^{\frac{3}{2}}}$.

$\int_0^1 \frac{1}{t^{\frac{3}{2}}} dt = \lim_{t \to 0^+} -2 + \frac{2}{t^{\frac{1}{2}}} = \infty$, so divergent.

5. For $0 < x \le 1$, $\sin^2 x \le 1 \rightarrow \frac{\sin^2 x}{\sqrt{x}} \le \frac{1}{\sqrt{x}}$.

$\int_0^1 \frac{1}{\sqrt{x}} dx = \lim_{t \to 0^+} 2 - 2t^{\frac{1}{2}} = 2 < \infty$, so convergent.

6. $\sin^2 x \le 1 \rightarrow \frac{\sin^2 x}{x^2} \le \frac{1}{x^2}$.

$\int_{-\infty}^{-1} \frac{1}{x^2} = \lim_{t \to -\infty} \left[\frac{-1}{x}\right]_t^{-1} = \lim_{t \to -\infty} 1 + \frac{1}{t} = 1 < \infty$

so convergent.

7. $-1 \le \sin x \le 1$

$1 \le \sin x + 2 \le 3$

$\rightarrow \frac{\sin x + 2}{x} \ge \frac{1}{x}$, and $\int_1^\infty \frac{1}{x} dx = \lim_{t \to \infty} \ln t = \infty$,

so divergent.

8. As in #7, $\frac{\sin x + 2}{x^2} \ge \frac{1}{x^2}$, and $\int_0^1 \frac{1}{x^2} dx = \lim_{t \to 0^+} \left[\frac{-1}{x}\right]_t^1$

$= \lim_{t \to 0^+} -1 + \frac{1}{t} = \infty$, so

divergent.

4. $\sqrt{1+\frac{1}{x^4}} \le \sqrt{\frac{1}{x^4}} = \frac{1}{x^2}$,

So $$\int_1^\infty \frac{1}{x}\sqrt{1+\frac{1}{x^4}}\,dx \le \int_1^\infty \frac{1}{x}\cdot\frac{1}{x^2}\,dx = \int_1^\infty \frac{1}{x^3}\,dx$$

$$= \lim_{t\to\infty}\left[\frac{-1}{2x^2}\right]_1^t = \lim_{t\to\infty} \frac{-1}{2t^2} + \frac{1}{2} = \frac{1}{2} < \infty$$

So convergent.

5. See the improper integrals worksheet solutions.