(A) (1) $\int_1^{\infty} \frac{1}{x}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x}\,dx = \lim_{t\to\infty} \left[\ln|x|\right]_1^t = \lim_{t\to\infty} \ln|t| = \infty$

$\rightarrow$ $\boxed{\text{diverges to } \infty.}$

(2) $\int_1^{\infty} \frac{1}{x^2}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^2}\,dx = \lim_{t\to\infty} \left[\frac{-1}{x}\right]_1^t = \lim_{t\to\infty} \frac{-1}{t} + 1 = 1$

$\rightarrow$ $\boxed{\text{Converges to } 1}$

(3) $\int_{-\infty}^{\infty} xe^{-x^2}\,dx = \lim_{t\to-\infty} \int_t^0 xe^{-x^2}\,dx + \lim_{t\to\infty} \int_0^t xe^{-x^2}\,dx$

$= \lim_{t\to-\infty} \left[\frac{-1}{2}e^{-x^2}\right]_t^0 + \lim_{t\to\infty} \left[\frac{-1}{2}e^{-x^2}\right]_0^t \quad \left(\begin{array}{c} u=-x^2 \\ -\frac{1}{2}du = x\,dx \end{array}\right)$

$= \lim_{t\to-\infty} \frac{-1}{2} + \frac{1}{2}e^{-t^2} + \lim_{t\to\infty} \frac{-1}{2}e^{-t^2} + \frac{1}{2} = \frac{-1}{2} + \frac{1}{2} = 0$

$\rightarrow$ $\boxed{\text{converges to } 0}$

(4) $\int_{-\infty}^{\infty} x\,dx = \lim_{t\to-\infty} \int_t^0 x\,dx + \lim_{t\to\infty} \int_0^t x\,dx$

$= \lim_{t\to-\infty} \left[\frac{1}{2}x^2\right]_t^0 + \lim_{t\to\infty} \left[\frac{1}{2}x^2\right]_0^t$

$= \lim_{t\to-\infty} \frac{-1}{2}t^2 + \lim_{t\to\infty} \frac{1}{2}t^2 = -\infty + \infty \ (\neq 0)$

Both of these integrals diverge, so $\boxed{\int_{-\infty}^{\infty} x\,dx \text{ diverges.}}$

(5) $\int_0^2 \frac{1}{x-1}\,dx = \lim_{t\to1^-} \int_0^t \frac{1}{x-1}\,dx + \lim_{t\to1^+} \int_t^2 \frac{1}{x-1}\,dx$

$= \lim_{t\to1^-} \left[\ln|x-1|\right]_0^t + \lim_{t\to1^+} \left[\ln|x-1|\right]_t^2$

$= \lim_{t\to1^-} \ln|t-1| + \lim_{t\to1^+} -\ln|t-1| = -\infty + \infty \ (\neq 0)$

Both of these integrals diverges. so $\boxed{\int_0^2 \frac{1}{x-1}\,dx \text{ diverges.}}$

(6) $\int_0^\infty x^3 e^{-x^2}\,dx$.

First $\int x^3 e^{-x^2}\,dx = \frac{1}{2}\int we^{w}\,dw \quad = \frac{1}{2}\left[we^{w} - e^{w}\right] + C$

$\left(\begin{array}{l} w = -x^2 \to -w = x^2 \\ -\frac{1}{2}w = x\,dx \end{array}\right) \qquad \left(\begin{array}{ll} u = w & v = e^w \\ du = dw & dv = e^w\,dw \end{array}\right)$

$= \frac{1}{2}\left[-x^2 e^{-x^2} - e^{-x^2}\right] + C$

So $\int_0^\infty x^3 e^{-x^2}\,dx = \lim_{t\to\infty}\int_0^t x^3 e^{-x^2}\,dx = \frac{1}{2}\lim_{t\to\infty}\left[-x^2 e^{-x^2} - e^{-x^2}\right]_0^t$

$= \frac{1}{2}\lim_{t\to\infty} -t^2 e^{-t^2} - e^{-t^2} + 1$

$\lim_{t\to\infty} -t^2 e^{-t^2} = \lim_{t\to\infty} \frac{-t^2}{e^{t^2}} \quad \left(\frac{-\infty}{\infty}\right)$

l'Hopital $= \lim_{t\to\infty} \frac{-\cancel{2t}}{\cancel{2t}e^{t^2}} = 0$

So $= \frac{1}{2}(0 + 0 + 1) = \frac{1}{2}$

So $\int_0^\infty x^3 e^{-x^2}\,dx$ converges to $\frac{1}{2}$.

(7) $\int_0^1 \frac{1}{x}\,dx = \lim_{t\to 0^+}\int_t^1 \frac{1}{x}\,dx = \lim_{t\to 0^+}\left[\ln|x|\right]_t^1 = \lim_{t\to 0^+} -\ln|t| = \infty$

So $\int_0^1 \frac{1}{x}\,dx$ diverges to ∞.

(8) $\int_0^1 \frac{1}{x(\ln x)^2}\,dx \quad = \int \frac{1}{u^2}\,du = \frac{-1}{u} + C = \frac{-1}{\ln t} + C$

$\left(\begin{array}{l} u = \ln x \\ du = \frac{1}{x}\,dx \end{array}\right)$

So $\int_0^1 \frac{1}{x(\ln x)^2}\,dx = \lim_{t\to 0^+}\left[\frac{-1}{\ln x}\right]_t^{\frac{1}{2}} + \lim_{t\to 1^-}\left[\frac{-1}{\ln x}\right]_{\frac{1}{2}}^t$

$= \lim_{t\to 0^+} \frac{-1}{\ln\frac{1}{2}} + \frac{1}{\ln t} + \lim_{t\to 1^-} \frac{-1}{\ln t} + \frac{1}{\ln\frac{1}{2}}$

$= \frac{-1}{\ln\frac{1}{2}} + 0 \quad + \infty + \frac{1}{\ln\frac{1}{2}} = +\infty$, so $\boxed{\text{diverges to } +\infty}$

(9) $\int_0^1 x^2 \ln x \, dx = \frac{1}{3}x^3 \ln x - \frac{1}{3}\int x^2 \, dx = \frac{1}{3}x^3 \ln x - \frac{1}{9}x^3 + C$

$u = \cancel{x}\ln x \qquad v = \frac{1}{3}x^3$

$du = \frac{1}{x}dx \qquad dv = x^2 \, dx$

So $\int_0^1 x^2 \ln x \, dx = \lim_{t \to 0^+} \left[\frac{1}{3}x^3 \ln x - \frac{1}{9}x^3\right]_t^1$

$= \lim_{t \to 0^+} -\frac{1}{9} - \frac{1}{3}t^3 \ln t + \frac{1}{9}t^3$

$= -\frac{1}{9} - 0 + 0 = -\frac{1}{9}$, so $\boxed{\int_0^1 x^2 \ln x \, dx = -\frac{1}{9}}$

(Since $\lim_{t \to 0^+} t^3 \ln t = \lim_{t \to 0^+} \frac{\ln t}{\frac{1}{t^3}}$ $\left(\frac{-\infty}{\infty}\right)$

l'Hopital $= \lim_{t \to 0^+} \frac{\frac{1}{t}}{\frac{-3}{t^4}} = \lim_{t \to 0^+} \frac{t^3}{-3} = 0$)

(10) $\int \frac{\ln x}{x} dx = \int u \, du \left(\begin{array}{l} u = \ln x \\ du = \frac{1}{x}dx \end{array}\right) = \frac{1}{2}u^2 + C = \frac{1}{2}(\ln x)^2 + C$

So $\int_0^1 \frac{\ln x}{x} dx = \frac{1}{2}\lim_{t \to 0^+} \left[(\ln x)^2\right]_t^1 = \frac{1}{2}\lim_{t \to 0^+} -(\ln t)^2 = -\infty$

so $\boxed{\int_0^1 \frac{\ln x}{x} dx \text{ diverges to } -\infty}$

(B) (1) $\int_{-10}^{10} \frac{1}{x^3+1} dx$ is improper (~~since~~ discontinuous at $x=-1$)

(2) $\int_0^\pi \tan x \, dx$ is improper (Discontinuous at $x = \frac{\pi}{2}$)

(3) $\int_{-1}^1 \frac{1}{x^2-x-2} dx$ is Improper (discontinuous at $x=-1$)

(4) $\int_0^2 \arctan x \, dx$ is Proper.

(5) $\int_1^{10} \ln|x-5| \, dx$ is Improper (discontinuous at $x=5$)

(C) (1) $\int_1^\infty \frac{x}{x^3+1}\,dx \le \int_1^\infty \frac{x}{x^3}\,dx = \int_1^\infty \frac{1}{x^2}\,dx < \infty$

So $\int_1^\infty \frac{x}{x^3+1}\,dx$ converges.

(2) $\int_1^\infty e^{-x^2}\,dx \le \int_1^\infty e^{-x}\,dx = \lim_{t\to\infty}\left[-e^{-x}\right]_1^t = \lim_{t\to\infty} -e^{-t}+e^{-1} = e^{-1} < \infty$

(since for $x \ge 1$, $x^2 \ge x \implies -x^2 \le -x \implies e^{-x^2} \le e^{-x}$)

so $\int_1^\infty e^{-x^2}\,dx$ converges.

(3) $\int_1^\infty \frac{1+e^{-x}}{x} \ge \int_1^\infty \frac{1}{x}\,dx = \infty$

So $\int_1^\infty \frac{1+e^{-x}}{x}\,dx$ diverges to ∞

(D) Case 1: $p=1$ We saw (A #7) that $\int_0^1 \frac{1}{x}\,dx$ diverges.

Case 2: $p \ne 1$: $\int_0^1 \frac{1}{x^p}\,dx = \lim_{t\to 0^+}\left[\frac{x^{-p+1}}{-p+1}\right]_t^1$

$$= \lim_{t\to 0^+} \frac{1}{-p+1} - \frac{t^{-p+1}}{-p+1}$$

Case 2a: $p > 1$: Then $-p<-1$, so $-p+1<0$ and $p-1>0$.

Then $\lim_{t\to 0^+} \frac{1}{-p+1} - \frac{t^{-p+1}}{-p+1} = \lim_{t\to 0^+} \frac{1}{-p+1} - \frac{1}{(-p+1)t^{p-1}} = \infty$

Case 2b: $p<1$: Then $-p>-1$, so $-p+1>0$.

Then $\lim_{t\to 0^+} \frac{1}{-p+1} - \frac{t^{-p+1}}{-p+1} = \frac{1}{-p+1} - 0 = \frac{1}{-p+1}$

So $\int_0^1 \frac{1}{x^p}\,dx$ converges only when $p<1$, and in that case its value is $\frac{1}{-p+1}$.