Definite integral: Suppose $f(x)$ is continuous on $[a,b]$. Divide $[a,b]$ into subintervals of length $\Delta x = \frac{b-a}{n}$ and choose a sample point x_i^* from each subinterval. Then

$$\int_a^b f(x)\ \mathrm{d}x = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x = \lim_{n\to\infty}\Delta x\sum_{i=1}^{n} f(x_i^*).$$

To compute integrals using this definition, it is standard to take the sample point x_i^* to be the right endpoint, $x_i^* = a + i\Delta x$.

Approximating integrals with Riemann sums: Choose n = number of rectangles and choose a sample point x_i^* from each subinterval. Then

$$\int_a^b f(x)\ \mathrm{d}x \approx \sum_{i=1}^{n} f(x_i^*)\Delta x = \Delta x\big[f(x_1^*) + f(x_2^*) + ... + f(x_n^*)\big] = A_n, \text{ where } \Delta x = \tfrac{b-a}{n}.$$

Each A_n is called a **Riemann sum**.

Antiderivative: An anti-derivative of $f(x)$ is a function $F(x)$ such that $F' = f$.

Indefinite integral: $\int f(x)\ \mathrm{d}x = F(x) + C$, where F is an anti-derivative of f.

FTC (”integration and differentiation are inverse processes”)

Part 1: $\frac{d}{dx}\int_a^x f(t)\ \mathrm{d}t = f(x)$.

Know how to apply the chain rule with part 1!

Part 2: $\int_a^b F'(x)\ \mathrm{d}x = F(b) - F(a)$

Main application of FTC2: the integral of the rate of change of $F(x)$ is the net change in $F(x)$ from $x = a$ to $x = b$.

eg, if $v(t)$ = velocity, then $\int_{t_1}^{t_2} v(t)\ \mathrm{d}t$ = net distance traveled = net change in position from time t_1 to t_2 (*not* total distance traveled (in general)).

Cauchy-Schwarz inequality: $\left(\int_a^b f(x)g(x)\ \mathrm{d}x\right)^2 \le \left(\int_a^b f(x)^2\ \mathrm{d}x\right)\left(\int_a^b g(x)^2\ \mathrm{d}x\right)$

Integration techniques

u-substitution/change of variables - undoing the chain rule: .

Given $\int_a^b f(g(x))g'(x)\ \mathrm{d}x$, substitute $u = g(x) \Rightarrow du = g'(x)\ \mathrm{d}x$ to convert $\int_a^b f(g(x))g'(x)\ \mathrm{d}x = \int_{g(a)}^{g(b)} f(u)\ \mathrm{d}u$.

u-substitution works for integrating compositions of functions; pick u to be the ’inside’ function (for indefinite integrals, drop the limits of integration).

Integration by parts - undoing the product rule: $\int u\ \mathrm{d}v = uv - \int v\ \mathrm{d}u$.

Generally, picking u in this descending order works, and $\mathrm{d}v$ is what’s left:

Inverse trig
Logarithm
Algebraic (polynomial)
Trig
Exponential

Summation formulas: In using the limit definition to compute definite integrals, these formulas may be helpful:

(1) $\displaystyle\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i.$

(2) $\displaystyle\sum_{i=1}^{n}(a_i \pm b_i) = \sum_{i=1}^{n} a_i \pm \sum_{i=1}^{n} b_i.$

(3) $\displaystyle\sum_{i=1}^{n} c = cn.$

(4) $\displaystyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2}.$

(5) $\displaystyle\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.$

(6) $\displaystyle\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2.$