MATH 3B Worksheet: u-substitution and integration by parts
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u-substitution/change of variables - undoing the chain rule:
Given $\int_a^b f(g(x))g'(x)\ dx$, substitute $u = g(x) \Rightarrow du = g'(x)\ dx$ to convert
$\int_a^b f(g(x))g'(x)\ dx = \int_{g(a)}^{g(b)} f(u)\ du$.
u-substitution works for integrating compositions of functions; pick u to be the 'inside' function (for indefinite integrals, drop the limits of integration).

1. Compute:

   (a) $\int 2x\ \sin(x^2)\ dx$.

   (b) $\int x^2 \left(3 - 5x^3\right)^4\ dx$.

   (c) $\int_{-\pi}^{\pi} \cos(x)\sin^{10}(x)\ dx$.

   (d) $\int \frac{\cos(\sqrt{x})}{\sqrt{x}}\ dx$.

   (e) $\int_2^e \frac{1}{x\ln(x)}\ dx$.

(a) $u = x^2 \rightarrow du = 2x\ dx \rightsquigarrow \int \sin(u)\ du = -\cos(u) + C$
$= -\cos(x^2) + C$

(b) $u = 3 - 5x^3 \rightarrow du = -15x^2 dx \rightarrow -\frac{1}{15} du = x^2 dx$
$-\frac{1}{15}\int u^4\ du = \frac{-1}{15} \cdot \frac{1}{5} u^5 + C = \frac{-1}{75}(3 - 5x^3)^5 + C$

(c) $u = \sin(x) \rightarrow du = \cos x\ dx \rightsquigarrow \int_0^0 u^{10}\ du = 0$ (or note right away the integrand is odd)

(d) $u = \sqrt{x} \rightarrow du = \frac{1}{2\sqrt{x}} dx \rightarrow 2du = \frac{1}{\sqrt{x}} dx$
$2\int \cos u\ du = 2\sin u + C = 2\sin(\sqrt{x}) + C.$

(e) $u = \ln x \rightarrow du = \frac{1}{x} dx \rightsquigarrow \int_{\ln 2}^{1} \frac{1}{u}\ du = \left[\ln|u|\right]_{\ln 2}^{1}$
$= \ln 1 - \ln(\ln 2) = -\ln(\ln 2).$

Integration by parts - undoing the product rule: $\int u\, dv = uv - \int v\, du$.
Generally, picking u in this descending order works, and dv is what's left:
Inverse trig
Logarithm
Algebraic (polynomial)
Trig
Exponential

2. Compute:

   (a) $\displaystyle\int xe^x\, dx$.

   (b) $\displaystyle\int x \sin(x)\, dx$.

   (c) $\displaystyle\int_1^e \ln(x)\, dx$.

   (d) $\displaystyle\int e^x \cos(x)\, dx$.

   (e) $\displaystyle\int \sin(\sqrt{x})\, dx$. (hint: first make a substitution)

(a) $u = x \quad v = e^x$
$du = dx \quad dv = e^x\,dx$
$xe^x - \int e^x\,dx = xe^x - e^x + C.$

(b) $u = x \quad v = -\cos x$
$du = dx \quad dv = \sin x\,dx$
$-x\cos x + \int \cos x\,dx = -x\cos x + \sin x + C.$

(c) $u = \ln x \quad v = x$
$du = \frac{1}{x}dx \quad dv = dx$
$\left[x \ln x\right]_1^e - \int_1^e dx = \left[x \ln x - x\right]_1^e$
$= (e(1) - e) - (1(0) - 1) = 1.$

(d) $u = \cos x \quad v = e^x$
$du = -\sin x\,dx \quad dv = e^x\,dx$
$\int e^x \cos x\,dx = e^x \cos x + \int e^x \sin x\,dx$ ← by parts again:
$u = \sin x \quad v = e^x$
$du = \cos x\,dx \quad dv = e^x\,dx$

$\rightarrow \int e^x \cos x\,dx = e^x \cos x + e^x \sin x - \int e^x \cos x\,dx$
$+\int e^x \cos x\,dx \qquad\qquad +\int e^x \cos x\,dx$

$2\int e^x \cos x\,dx = e^x \cos x + e^x \sin x + c \rightarrow \int e^x \cos x\,dx = \dfrac{e^x(\cos x + \sin x)}{2} + C.$

(e) $w = \sqrt{x} \rightarrow dw = \frac{1}{2\sqrt{x}}dx \rightarrow dx = 2\sqrt{x}\,dw = 2w\,dw$:
$\rightsquigarrow 2\int w \sin(w)\,dw = 2[-w\cos w + \sin w] + C$ (same as part (b))
$= 2[-\sqrt{x}\cos\sqrt{x} + \sin\sqrt{x}] + C.$

MATH 3B Worksheet: Riemann sums and definite integrals

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1. Consider the integral $\int_0^4 16 - t^2\, dt$. Find the Riemann sum for this integral using:

   (a) right-hand sums for $n = 4$.

   (b) left-hand sums for $n = 4$.

   (c) Sketch a graph for (a)-(b). Which underestimates the actual value of the integral? Overestimates? What can you say in general?

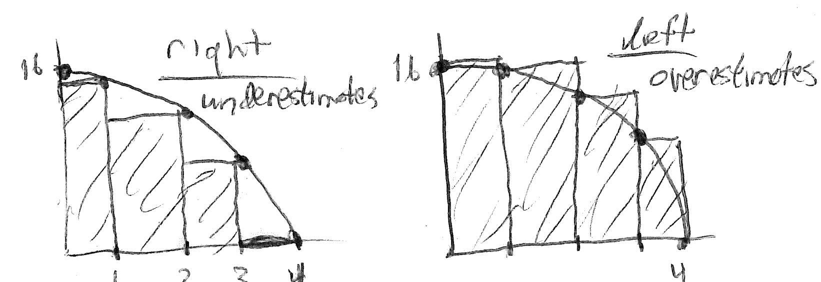
   (d) Now suppose that $v(t) = 16 - t^2$ is the velocity (in ft/s) of a car after t seconds, so the car comes to a stop after 4 seconds. Suppose that a kitten is 45 feet in front of the car at $t = 0$. Based on (a)-(c), can we determine whether or not the car struck the kitten?

   (e) Compute the actual value of the integral to find that the kitten lives to see another day.

(a) $\Delta t = \frac{4-0}{4} = 1$ so $A_4 = (16-1^2) + (16-2^2) + (16-3^2) + (16-4^2) = 34.$

(b) Shift left one: $(16-0^2) + (16-1^2) + (16-2^2) + (16-3^2) = 50.$

(c)



In general, right hand will give underestimate for decreasing function.

" " " ~~right~~ " " " " overestimate " increasing "

" " " left " " " " underestimate " increasing "

" " " left " " " " overestimate " decreasing "

(d) All we know is that it took at least 34 ft and at most 50 ft for the car to stop, so we can't determine anything based on (a) - (c).

(e) $\int_0^4 16 - t^2\, dt = \left[16t - \frac{1}{3}t^3\right]_0^4 = 16(4) - \frac{1}{3}4^3 = 42\frac{2}{3}$ ft,

= distance it takes car to stop,

So the kitten lives!

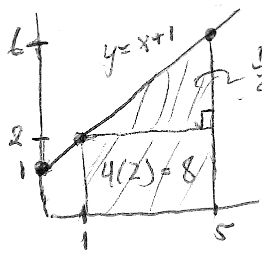
2. Evaluate the following integrals by interpreting them in terms of areas:

(a) $\int_1^5 x+1\ dx$.

(b) $\int_{-2}^{2} |2x+2|\ dx$.

(c) $\int_{2}^{6} -\sqrt{4-(x-4)^2}\ dx$.

(a)

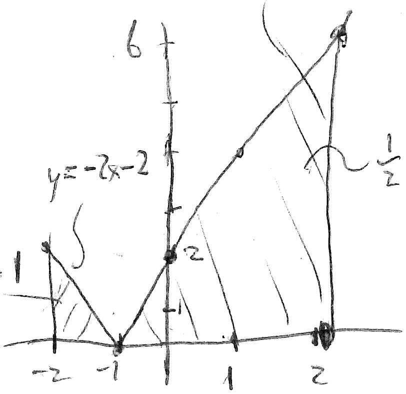


$\frac{1}{2}(4)(4)=8$

$4(2)=8$

$8+8=16$

(b)



$\frac{1}{2}(1)(2)=1$

$\frac{1}{2}(3)(6)=9$

$9+1=10$

(c) $(x-a)^2+(y-b)^2=r^2$ is a circle with radius r and center (a,b) so $y=-\sqrt{4-(x-4)^2}$ is the bottom half of a circle with radius 2 and center $(4,0)$. $A=-\frac{1}{2}\pi(2^2)=-2\pi$

3. Use the limit definition of the definite integral to compute the following:

(a) $\int_0^2 2\ dx$.

(b) $\int_0^2 x\ dx$.

(c) $\int_1^2 x+1\ dx$.

(d) $\int_0^2 x^2+1\ dx$.

(a) $\Delta x=\frac{2-0}{n}=\frac{2}{n}$. $x_i=0+i\Delta x=\frac{2i}{n}$. $\lim_{n\to\infty}\frac{2}{n}\sum_{i=1}^{n}2=\lim_{n\to\infty}\frac{2}{n}\cdot 2n=4$.

$f(x_i)=2$

(b) $\Delta x=\frac{2}{n}$. $x_i=\frac{2i}{n}$. $f(x_i)=\frac{2i}{n}$. $\lim_{n\to\infty}\frac{2}{n}\sum_{i=1}^{n}\frac{2i}{n}=\lim_{n\to\infty}\frac{4}{n^2}\sum_{i=1}^{n}i$

$=\lim_{n\to\infty}\frac{4}{n^2}\frac{n(n+1)}{2}$

$=\lim_{n\to\infty}\frac{4n^2+4n}{2n^2}=\frac{4}{2}=2.$

(c) $\Delta x = \frac{2-1}{n} = \frac{1}{n}$. $x_i = 1 + i\Delta x = 1 + \frac{i}{n}$, $f(x_i) = x_i + 1 = 2 + \frac{i}{n}$.

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} 2 + \frac{i}{n} = \lim_{n\to\infty} \frac{1}{n}\left(\sum_{i=1}^{n} 2 + \sum_{i=1}^{n} \frac{i}{n}\right)$$

$$= \lim_{n\to\infty} \frac{1}{\cancel{n}}(2\cancel{n}) + \frac{1}{n^2}\sum_{i=1}^{n} i$$

$$= \lim_{n\to\infty} 2 + \frac{1}{n^2}\cdot\frac{n(n+1)}{2}$$

$$= \lim_{n\to\infty} 2 + \frac{n^2+n}{2n^2} = 2 + \frac{1}{2} = \frac{5}{2}.$$

(d) $\Delta x = \frac{2-0}{n} = \frac{2}{n}$. $x_i = \frac{2i}{n}$. $f(x_i) = x_i^2 + 1 = \left(\frac{2i}{n}\right)^2 + 1 = \frac{4i^2}{n^2} + 1$.

$$\lim_{n\to\infty} \frac{2}{n}\sum_{i=1}^{n} \frac{4i^2}{n^2} + 1 = \lim_{n\to\infty} \frac{2}{n}\left(\sum_{i=1}^{n} \frac{4i^2}{n^2} + \sum_{i=1}^{n} 1\right)$$

$$= \lim_{n\to\infty} \frac{2(4)}{n^3}\left(\sum_{i=1}^{n} i^2\right) + \frac{2}{n}n$$

$$= \lim_{n\to\infty} \frac{8}{n^3}\frac{n(n+1)(2n+1)}{6} + 2$$

$$= \lim_{n\to\infty} \frac{16n^3 + \text{stuff}}{6n^3} + 2 = \frac{16}{6} + 2$$

$$= \frac{14}{3}.$$

MATH 3B Worksheet: The Fundamental Theorem of Calculus
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Cauchy-Schwarz inequality: $\left(\int_a^b f(x)g(x)dx\right)^2 \leq \left(\int_a^b f(x)^2dx\right)\left(\int_a^b g(x)^2dx\right)$

1. Use the Cauchy-Schwarz inequality to show the following:

   (a) $\left(\int_0^1 (x+1)xdx\right)^2 \leq \frac{7}{9}$.

   (b) $\left(\int_0^1 x^2+2x+1dx\right)^2 \leq \frac{49}{9}$.

   (c) $\int_0^1 x^2-1dx \leq \frac{\sqrt{7}}{3}$.

   (d) $\left(\int_1^2 x^2dx\right)\left(\int_1^2 \frac{1}{x^2}dx\right) \geq$ 1.

(a) $\left(\int_0^1 (x+1)x\,dx\right)^2 \leq \left(\int_0^1 (x+1)^2dx\right)\left(\int_0^1 x^2\,dx\right)$

$= \left(\int_0^1 x^2+2x+1\,dx\right)\left(\int_0^1 x^2\,dx\right)$

$= \left[\frac{1}{3}x^3+x^2+x\right]_0^1 \cdot \left[\frac{1}{3}x^3\right]_0^1$

$= \left(\frac{1}{3}+1+1\right)\left(\frac{1}{3}\right) = \frac{7}{9}$.

(b) $\left(\int_0^1 (x+1)^2\,dx\right)^2 \leq \left(\int_0^1 (x+1)^2dx\right)\left(\int_0^1 (x+1)^2\,dx\right)$

$= \left(\frac{1}{3}+1+1\right)\left(\frac{1}{3}+1+1\right)$ (by (a))

$= \frac{7}{3}\cdot\frac{7}{3} = \frac{49}{9}$

(c) $$\left(\int_0^1 (x+1)(x-1)\,dx\right)^2 \le \left(\int_0^1 (x+1)^2\,dx\right)\left(\int_0^1 (x-1)^2\,dx\right)$$

$$= \left(\int_0^1 x^2+2x+1\,dx\right)\left(\int_0^1 x^2-2x+1\,dx\right)$$

$$= \left[\frac{1}{3}x^3 + x^2 + x\right]_0^1 \cdot \left[\frac{1}{3}x^3 - x^2 + x\right]_0^1$$

$$= \left(\frac{7}{3}\right)\left(\frac{1}{3}\right) = \frac{7}{9}.$$

Taking square roots of $\left(\int_0^1 (x+1)(x-1)\,dx\right)^2 \le \frac{7}{9}$

shows $\int_0^1 (x+1)(x-1) \le \sqrt{\frac{7}{9}} = \frac{\sqrt{7}}{3}$

(d) Here $f(x) = x$ and $g(x) = \frac{1}{x}$, so

$$\left(\int_1^2 x^2\,dx\right)\left(\int_1^2 \frac{1}{x^2}\,dx\right) \ge \left(\int_1^2 f(x)\,g(x)\,dx\right)^2$$

$$= \left(\int_1^2 x \cdot \frac{1}{x}\,dx\right)^2$$

$$= \left(\int_1^2 1\,dx\right)^2$$

$$= 1^2 = 1.$$