MATH 3B Notes: Integrating rational functions

A. To integrate a rational function $p(x)/q(x)$, like $\dfrac{x}{x^2+3x+2}$:

(1) If the degree of the numerator (highest power of x) is $\geq$ the degree of the denominator, first do long division.
(2) Factor the denominator into linear terms and irreducible quadratics.
(3) Get the partial fraction decomposition of $p(x)/q(x)$ using the rules:

factor in denominator		**term in partial fraction decomposition**
$(ax+b)^k$	$\Rightarrow$	$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \ldots + \frac{A_k}{(ax+b)^k}$
$(ax^2+bx+c)^k$	$\Rightarrow$	$\frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \ldots + \frac{A_kx+B_k}{(ax^2+bx+c)^k}$

(4) Set $p(x)/q(x)$ in factored form equal to its partial fraction decomposition, multiply by $q(x)$ and equate coefficients to solve for constants (or pick values of x to make constants 0).
(5) We just have to integrate expressions like $\frac{A}{(ax+b)^k}$ and $\frac{Ax+B}{(ax^2+bx+c)^k}$. The first, $\frac{A}{(ax+b)^k}$, is done with a simple substitution, $u = ax+b$.
(6) We'll break $\frac{Ax+B}{(ax^2+bx+c)^k}$ into two fractions to integrate it. For the first we'll let $u = ax^2+bx+c$, $\mathrm{d}u = (2ax+b)\mathrm{d}x$. We want the numerator of the first fraction to be a multiple of $\mathrm{d}u = (2ax+b)\mathrm{d}x$, which then becomes a simple integral.
(7) To integrate the second fraction, we'll complete the square and make a substitution to convert this to an integral like $\displaystyle\int \frac{du}{(u^2+1)^k}$.
(8) Next let $u = \tan t$, $\mathrm{d}u = (1+\tan^2 t)\mathrm{d}t$, to get an integral $\displaystyle\int \cos^m t\ \mathrm{d}t$.
(9) The power of cosine will be even, and you use the trig identity $\cos^2 t = \frac{1}{2}(1+\cos(2t))$ repeatedly to integrate. For example,

$$\int \cos^4 t\ \mathrm{d}t = \int \left[\frac{1}{2}(1+\cos(2t))\right]^2 \mathrm{d}t =$$

$$\int \frac{1}{4}\left[1 + 2\cos(2t) + \cos^2(2t)\right] \mathrm{d}t = \int \frac{1}{4}\left[1 + 2\cos(2t) + \tfrac{1}{2}(1+\cos(4t))\right] \mathrm{d}t.$$

B. To integrate a rational function $p(\cos x, \sin x)/q(\cos x, \sin x)$, like $\dfrac{\sin x}{1+ \cos x}$:

Method 1: Let $t = \tan(x/2)$ and use the formulas

$$\cos x = \frac{1-t^2}{1+t^2}, \sin x = \frac{2t}{1+t^2}, \text{ and } \mathrm{d}x = \frac{2}{1+t^2}\mathrm{d}t$$

to convert this to a usual rational function and see **A.**

Method 2 - Regles de Bioche: If $p/q\ \mathrm{d}x$ remains the same after the one of the following substitutions, make that substitution to convert the integral to a rational function like in case **A.**

(1) $x \mapsto -x$, $\mathrm{d}x \mapsto -\mathrm{d}x$, then let $t = \cos x$, $-\mathrm{d}t = \sin x\ \mathrm{d}x$.
(2) $x \mapsto \pi - x$, $\mathrm{d}x \mapsto -\mathrm{d}x$, then let $t = \sin x$, $\mathrm{d}t = \cos x\ \mathrm{d}x$.
(3) $x \mapsto \pi + x$, $\mathrm{d}x \mapsto \mathrm{d}x$, then let $t = \tan x$, $\mathrm{d}t = \sec^2 x\ \mathrm{d}x = (\tan^2 x + 1)\mathrm{d}x$.

To check the above conditions, use:

$$\sin(-x) = -\sin(x),\ \sin(\pi - x) = \sin(x),\ \sin(\pi + x) = -\sin(x)$$
$$\cos(-x) = \cos(x),\ \cos(\pi - x) = -\cos(x),\ \cos(\pi + x) = -\cos(x)$$

(if you speak French: http://fr.wikipedia.org/wiki/Regles_de_Bioche)

C. To integrate a rational function $p(e^x)/q(e^x)$, like $\dfrac{1+e^x}{2+e^{2x}}$:

Substitute $u = e^x$, $\mathrm{d}u = e^x\ \mathrm{d}x$, so $\mathrm{d}x = \mathrm{d}u/u$, to convert this to a rational function like in case **A.**