① $\int \frac{\sin x}{3+\sin^2 x}\,dx.$

Since $x \mapsto -x$, $dx \mapsto -dx$ gives $\frac{\sin(-x)}{3+(\sin(-x))^2}(-dx) = \frac{-\sin x\,(-dx)}{3+(-\sin x)^2}$

$$= \frac{\sin x}{3+\sin^2 x}\,dx$$

we let $t = \cos x$, $-dt = \sin x\,dx$. to get

$$-\int \frac{1}{3+1-t^2}\,dt$$

since $t = \cos x$

$\to t^2 = \cos^2 x = 1-\sin^2 x$

$\to \sin^2 x = 1-t^2$

$$= -\int \frac{1}{4-t^2}\,dt$$

$$= \int \frac{1}{t^2-4}\,dt.$$

$$\frac{1}{t^2-4} = \frac{1}{(t-2)(t+2)} = \frac{A}{t-2} + \frac{B}{t+2}$$

$\to \quad 1 = A(t+2) + B(t-2)$

$t=2: \quad 1 = 4A \to A = \frac{1}{4}$

$t=-2: \quad 1 = -4B \to B = \frac{-1}{4}$

$$\int \frac{\frac{1}{4}}{t-2} + \frac{\frac{-1}{4}}{t+2}\,dt = \frac{1}{4}\ln|t-2| - \frac{1}{4}\ln|t+2| + C$$

$$= \boxed{\frac{1}{4}\ln|\cos(x)-2| - \frac{1}{4}\ln|\cos(x)+2| + C}$$

② $\displaystyle\int \frac{\cos x}{\sin^3 x - \sin x}\, dx$

Since $\begin{array}{l} x \longmapsto \pi - x \\ dx \longmapsto -dt \end{array}$ gives $\dfrac{\cos(\pi - x) \; (-dx)}{(\sin \pi - x)^3 - \sin \pi - x} = \dfrac{-\cos x \; (-dx)}{\sin^3 x - \sin x}$.

$$= \frac{\cos x}{\sin^3 x - \sin x}\, dx$$

we let $t = \sin x$, $dt = \cos x \, dx$ to get

$\displaystyle\int \frac{1}{t^3 - t}\, dt$. Then $\dfrac{1}{t^3 - t} = \dfrac{1}{t(t^2-1)}$

$$= \frac{1}{t(t+1)(t-1)} = \frac{A}{t} + \frac{B}{t+1} + \frac{C}{t-1}$$

$$\rightarrow \quad 1 = A(t+1)(t-1) + Bt(t-1) + Ct(t+1)$$

$t = 0 : \; 1 = -A \rightarrow A = -1$

$t = -1 : \; 1 = 2B \rightarrow B = \frac{1}{2}$

$t = 1 : \; 1 = 2C \rightarrow C = \frac{1}{2}$

$$\int \frac{-1}{t} + \frac{\frac{1}{2}}{t+1} + \frac{\frac{1}{2}}{t-1}\, dt = -\ln|t| + \tfrac{1}{2}\ln|t+1| + \tfrac{1}{2}\ln|t-1| + C$$

$$= \boxed{-\ln|\sin x| + \tfrac{1}{2}\ln|\sin(x)+1| + \tfrac{1}{2}\ln|\sin(x)-1| + C}$$

③ $\int \frac{\sin^2 x}{\cos^2 x+1} \sec^2 x \, dx$

Since $x \mapsto \pi + x$ gives $\frac{(\sin \pi + x)^2}{(\cos \pi + x)^2 + 1} (\sec(\pi + x))^2 \, dx$
$dx \mapsto dx$

$$= \frac{\sin^2 x}{\cos^2 x + 1} \sec^2 x \, dx$$

we let $t = \tan x$, $dt = \sec^2 x \, dx$ to get

$$t^2 = \tan^2 x = \sec^2 x - 1 = \frac{1}{\cos^2 x} - 1$$

$$\to \frac{1}{\cos^2 x} = t^2 + 1 \to \cos^2 x = \frac{1}{t^2+1},$$

Then $\sin^2 x = 1 - \cos^2 x = 1 - \frac{1}{t^2+1}$.

$$\int \frac{1 - \frac{1}{t^2+1}}{\frac{1}{t^2+1} + 1} dt = \int \frac{t^2+1-1}{t^2+1+1} dt = \int \frac{t^2}{t^2+2} dt$$

$$\begin{array}{r} 1 \\ t^2+2 \overline{) \; t^2 \quad} \\ \underline{t^2+2} \\ -2 \end{array} \longrightarrow = \int 1 + \frac{-2}{t^2+2} dt = t + \underbrace{\int \frac{-2}{t^2+2} dt}$$

let $t = \sqrt{2} \tan\theta$, $dt = \sqrt{2}\sec^2\theta \, d\theta$ to get

$$t + \int \frac{-\cancel{2}\sqrt{2} \sec^2\theta \, d\theta}{\cancel{2}\tan^2\theta + 2} = t + \int \frac{-\cancel{2}\sqrt{2} \sec^2\theta \, d\theta}{\cancel{2}(\tan^2\theta + 1)} = t + \int \frac{-\sqrt{2} \cancel{\sec^2\theta} \, d\theta}{\cancel{\sec^2\theta}}$$

$$= t - \sqrt{2} \int d\theta = t - \sqrt{2}\,\theta = t - \sqrt{2} \arctan\left(\frac{t}{\sqrt{2}}\right) + C$$

since $t = \sqrt{2}\tan\theta \to \tan\theta = \frac{t}{\sqrt{2}} \Rightarrow \theta = \arctan\left(\frac{t}{\sqrt{2}}\right)$

③ con'td

$$\boxed{= \tan x - \sqrt{2}\arctan\left(\frac{\tan x}{\sqrt{2}}\right) + C}$$

④ $$\int_{\pi/2}^{\pi/3} \frac{1}{1+\sin x - \cos x}\,dx$$

Broche doesn't work (merde!), so let $t = \tan\frac{x}{2}$

and use $\cos x = \frac{1-t^2}{1+t^2}$ $\sin x = \frac{2t}{1+t^2}$ $dx = \frac{2}{1+t^2}\,dt$

$$\int_1^{\frac{1}{\sqrt{3}}} \frac{2}{(1+t^2)\left(1+\frac{2t}{1+t^2}-\frac{1-t^2}{1+t^2}\right)}\,dt = \int_1^{\frac{1}{\sqrt{3}}} \frac{2}{1+t^2+2t-1+t^2}\,dt$$

$$= 2\int_1 \frac{1}{2t^2+2t}\,dt = \int_1^{\frac{1}{\sqrt{3}}} \frac{1}{t(t+1)}\,dt$$

(bounds: $x = \frac{\pi}{2} \rightarrow t = \tan\frac{\frac{\pi}{2}}{2} = \tan\frac{\pi}{4} = 1$

$x = \frac{\pi}{3} \rightarrow t = \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$.

$$\frac{1}{t(t+1)} = \frac{A}{t} + \frac{B}{t+1} \rightarrow 1 = A(t+1) + Bt$$

$t=0: 1=A$ $t=-1: 1=-B \rightarrow B=-1$

$$\int_1^{\frac{1}{\sqrt{3}}} \frac{1}{t} + \frac{-1}{t+1}\,dt = \Big[\ln|t| - \ln|t+1|\Big]_1^{\frac{1}{\sqrt{3}}}$$

$$\boxed{\begin{aligned}&= \left(\ln\left|\tfrac{1}{\sqrt{3}}\right| - \ln\left|\tfrac{1}{\sqrt{3}}+1\right|\right)\\ &- \left(\ln|1| - \ln|2|\right)\end{aligned}}$$

⑤ $\int \frac{1}{1-\cos x}\,dx$

Bioche rules don't apply (zut!) so let $t = \tan\frac{x}{2}$ and use

$$\cos x = \frac{1-t^2}{1+t^2} \quad ; \quad dx = \frac{2}{1+t^2}\,dt$$

$$\int \frac{2}{(1+t^2)\left(1-\frac{1-t^2}{1+t^2}\right)}\,dt = \int \frac{2}{1+t^2-1+t^2}\,dt = \cancel{2}\int \frac{1}{\cancel{2}t^2}\,dt$$

$$= \int t^{-2}\,dt$$

$$= \frac{-1}{t} + C$$

$$= \boxed{\frac{-1}{\tan\frac{x}{2}} + C}$$

$$= \boxed{-\cot\frac{x}{2} + C}$$

⑥ $\int \frac{x^4}{x-1}\,dx$

$$\begin{array}{r|l} & x^3+x^2+x+1 \\ \hline (x-1) & x^4 \\ & \underline{x^4-x^3} \\ & x^3 \\ & \underline{x^3-x^2} \\ & x^2 \\ & \underline{x^2-x} \\ & x \\ & \underline{x-1} \\ & ① \end{array}$$

$$= \int x^3+x^2+x+1+\frac{1}{x-1}\,dx$$

$$= \boxed{\frac{1}{4}x^4+\frac{1}{3}x^3+\frac{1}{2}x^2+x+\ln|x-1|+C}$$

⑦ $\int \frac{x+4}{x^2+4x+4}\,dx$

$$\frac{x+4}{(x+2)^2} = \frac{A}{x+2} + \frac{B}{(x+2)^2}$$

$$\underline{1}x + \underline{4} = A(x+2) + B = \underline{A}x + \underline{\underline{2A+B}}$$

$A = 1$

$2A+B = 4 \rightarrow 2+B = 4 \rightarrow B = 2$

$$\int \frac{1}{x+2} + \frac{2}{(x+2)^2}\,dx = \boxed{\ln|x+2| - \frac{2}{x+2} + C}$$

⑧ $\int_1^2 \frac{1}{x(x^2+4)^2}\,dx$

$$\frac{1}{x(x^2+4)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+4} + \frac{Dx+E}{(x^2+4)^2}$$

$$1 = A(x^2+4)^2 + (Bx+C)x(x^2+4) + (Dx+E)x$$

$x=0$: $1 = 16A \rightarrow A = \frac{1}{16}$

$$1 = \frac{1}{16}(x^4+8x^2+16) + (Bx^2+Cx)(x^2+4) + Dx^2 + Ex$$

$$1 = \frac{1}{16}x^4 + \frac{1}{2}x^2 + 1 + Bx^4 + Cx^3 + 4Bx^2 + 4Cx + Dx^2 + Ex$$

$$1 = (\tfrac{1}{16}+B)x^4 + Cx^3 + (\tfrac{1}{2}+4B+D)x^2 + (4C+E)x + 1$$

$B + \frac{1}{16} = 0 \rightarrow B = \frac{-1}{16}$, $C = 0$, $\frac{1}{2} + 4B + D = 0 \rightarrow D = \frac{-1}{4}$, $4C+E=0 \rightarrow E = 0$

$$\rightarrow \int_1^2 \frac{\frac{1}{16}}{x} + \frac{\frac{-1}{16}x}{x^2+4} + \frac{\frac{-1}{4}x}{(x^2+4)^2}\,dx = \Big[\frac{1}{16}\ln|x| - \frac{1}{16}\cdot\frac{1}{2}\ln|x^2+4| - \frac{1}{4}\left(\frac{-1}{2}\right)\frac{1}{x^2+4}\Big]_1^2$$

$$= \boxed{\Big[\frac{1}{16}\ln|x| - \frac{1}{32}\ln(x^2+4) + \frac{1}{8(x^2+4)}\Big]_1^2}$$

$$= \boxed{\frac{1}{16}\ln 2 - \frac{1}{32}\ln(8) + \frac{1}{64} + \frac{1}{32}\ln(5) - \frac{1}{40}}$$

$$⑨\ \int \frac{x^2-3x+7}{(x^2-4x+6)^2}\,dx$$

$$\frac{x^2-3x+7}{(x^2-4x+6)^2} = \frac{Ax+B}{x^2-4x+6} + \frac{Cx+D}{(x^2-4x+6)^2}$$

$$x^2-3x+7=(Ax+B)(x^2-4x+6)+Cx+D$$

$$x^2-3x+7=Ax^3+(-4A+B)x^2+(6A-4B+C)x+6B+D$$

$$\rightarrow A=0,\ -4A+B=1 \rightarrow B=1,\ 6A-4B+C=-3 \rightarrow C=1,\ 6B+D=7 \rightarrow D=1$$

$$\rightarrow \int \frac{1}{x^2-4x+6} + \frac{x+1}{(x^2-4x+6)^2}\,dx$$

$$= \underbrace{\int \frac{1}{(x-2)^2+2}\,dx}_{\text{Ⅰ}} + \underbrace{\int \frac{x-2}{(x^2-4x+6)^2}\,dx}_{\text{Ⅱ}} + \underbrace{\int \frac{3}{(x^2-4x+6)^2}\,dx}_{\text{Ⅲ}}$$

Ⅰ: $u=x-2$, $du=dx$

$$\int \frac{1}{u^2+2}\,du. \quad u=\sqrt{2}\tan\theta,\ du=\sqrt{2}\sec^2\theta\,d\theta \rightarrow \int \frac{\sqrt{2}\cancel{\sec^2\theta}\,d\theta}{2\cancel{\sec^2\theta}} = \frac{\sqrt{2}}{2}\theta = \frac{\sqrt{2}}{2}\arctan\left(\frac{u}{\sqrt{2}}\right)$$

$$\boxed{=\frac{\sqrt{2}}{2}\arctan\left(\frac{x-2}{\sqrt{2}}\right)}$$

Ⅱ: $u=x^2-4x+6$, $du=(2x-4)\,dx$, $\frac{1}{2}du=(x-2)\,dx$

$$=\frac{1}{2}\int \frac{1}{u^2}\,du = \frac{-1}{2u} \boxed{= \frac{-1}{2(x^2-4x+6)}}$$

Ⅲ:

$$3\int \frac{1}{((x-2)^2+2)^2}\,dx \overset{\substack{u=x-2\\du=dx}}{=} 3\int \frac{1}{(u^2+2)^2}\,du \overset{\substack{u=\sqrt{2}\tan\theta\\du=\sqrt{2}\sec^2\theta\,d\theta}}{=} 3\int \frac{\sqrt{2}\sec^{\cancel{2}}\theta\,d\theta}{4\sec^{\cancel{4}\,2}\theta}.$$

$$= \frac{3\sqrt{2}}{4}\int \cos^2\theta\,d\theta = \frac{3\sqrt{2}}{4}\int \frac{1}{2}(1+\cos 2\theta)\,d\theta = \frac{3\sqrt{2}}{4}\left[\frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\right]$$

$$= \frac{3\sqrt{2}}{8}\left(\arctan\left(\frac{x-2}{\sqrt{2}}\right) + \sin\theta\cos\theta\right)$$

$$= \frac{3\sqrt{2}}{8}\left(\arctan\left(\frac{x-2}{\sqrt{2}}\right) + \frac{x-2}{\sqrt{x^2-4x+6}}\,\frac{\sqrt{2}}{\sqrt{x^2-4x+6}}\right)$$

⑨ cont'd

Final: $\textcircled{I} + \textcircled{II} + \textcircled{III} + C$.

(note we completed the square:

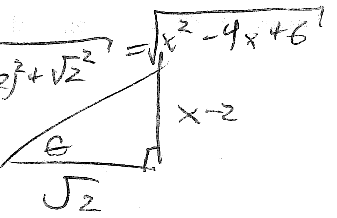
$$x^2 - 4x + 6 = \underbrace{x^2 - 4x + 4}\underbrace{- 4 + 6} = (x-2)^2 + 2$$

Use a trig identity for $\textcircled{III}$: $\sin 2\theta = 2\sin\theta\cos\theta$.

Used right triangle trig to get $\sin\theta$, $\cos\theta$ in terms of x.

$u = x-2 = \sqrt{2}\tan\theta$

$\rightarrow \tan\theta = \frac{x-2}{\sqrt{2}}$



$\sin\theta = \frac{opp}{hyp}$

$\cos\theta = \frac{adj}{hyp}$.

⑩ $$\int \frac{e^{2x}}{e^{2x} + 3e^x + 2}\,dx \qquad u = e^x \quad du = e^x dx \Rightarrow \frac{du}{e^x} = \frac{du}{u} = dx$$

$$= \int \frac{\cancel{u^2}}{\cancel{u}(u^2 + 3u + 2)}\,du = \int \frac{u}{(u+1)(u+2)}\,du$$

$$\frac{u}{(u+1)(u+2)} = + \frac{A}{u+1} + \frac{B}{u+2}$$

$$u = A(u+2) + B(u+1)$$

~~$u=0$: $0 = 2A \Rightarrow A = 0$~~ $u = -1$: $-1 = A$

~~$u=-1$: $1 = -B \Rightarrow B = -1$~~ $u = -2$: $-2 = -B \rightarrow B = 2$

~~$u=-2$: $4 = 2C \rightarrow C = 2$~~ $\rightarrow \int \frac{-1}{u+1} + \frac{2}{u+2}\,du$

~~$\Rightarrow \int$~~

$$= \boxed{-\ln|e^x+1| + 2\ln|e^x+2| + C}$$

$$\boxed{= \ln\frac{(e^x+2)^2}{e^x+1} + C}$$