MATH 3B Final practice problems

---

1. Compute the following:

   (a) $\int e^{-x} \cos(2x)\ \mathrm{d}x$

   Solution: Apply integration by parts twice and get $\int e^{-x} \cos(2x)\ \mathrm{d}x$ by itself.

   (b) $\int \frac{1+x^2}{\sqrt{3x+x^3}}\ \mathrm{d}x$

   Solution: Let $u = 3x + x^3$. Then $\frac{1}{3}du = (1 + x^2)dx$ and the rest is straightforward.

   (c) $\int \cos(\sqrt{x})\ \mathrm{d}x$

   Solution: Let $u = \sqrt{x}$. Then $du = \frac{1}{2\sqrt{x}}dx \Rightarrow 2\sqrt{x}du = 2udu = dx$. Then integration by parts.

   (d) $\frac{d}{dx}\int_x^{x^2} t^3\ \mathrm{d}t$

   Solution: By FTC1 and the chain rule,

   $$\frac{d}{dx}\int_x^{x^2} t^3 dt = \frac{d}{dx}\left[\int_0^{x^2} t^3 dt - \int_0^{x} t^3 dt\right] = (x^2)^3 2x - x^3 = 2x^7 - x^3.$$

   (e) $\int \frac{(ln(x))^3}{x}\ \mathrm{d}x$

   Solution: Let $u = ln(x)$, so $du = \frac{1}{x}dx$...

   (f) $\int \frac{x^3 - \sqrt{x}}{x^2}\ \mathrm{d}x$

   Solution: Write as two fractions and integrate: $\frac{x^3-\sqrt{x}}{x^2} = x - x^{-\frac{3}{2}}$.

   (g) $\int \sin^3(x)\cos^2(x)\ \mathrm{d}x$

   Solution: $\int \sin^3(x)\cos^2(x)\ \mathrm{d}x = \int (1 - \cos^2(x))\sin(x)\cos^2(x)\ \mathrm{d}x$.
   Let $u = \cos(x) \Rightarrow -du = \sin(x)dx$...

(h) $\displaystyle\int_0^1 \frac{4}{4x-1}\,\mathrm{d}x$

Solution: Notice that the integrand has a discontinuity at $x = 1/4$, so

$$\int_0^1 \frac{4}{4x-1}dx = \lim_{t\to 1/4^-} \int_0^t \frac{4}{4x-1}dx + \lim_{t\to 1/4^+} \int_t^1 \frac{4}{4x-1}dx.$$

If you show that one of these integrals diverge, then the whole thing does (both diverge).

(i) $\displaystyle\int \frac{1}{x^2-1}dx$

Solution: Factor and do partial fractions...

(j) $\displaystyle\int \tan^5(x)\sec^7(x)\,\mathrm{d}x$

Solution: Let $u = \sec x, du = \sec x \tan x$ and get the tangents in terms of secants.

(k) $\displaystyle\int \frac{\sqrt{9-x^2}}{x^2}\,\mathrm{d}x$

Solution: Let $x = 3\sin\theta$ and it should simplify to a familar integral.

(l) $\displaystyle\int \frac{x-4}{x^2-5x+6}\,\mathrm{d}x$

Solution: Factor and do partial fractions...

2. Show that the region $R = \{(x, y) : x \geq 1, 0 \leq y \leq 1/x\}$ has infinite area.

Solution: The area of the region R is

$$\int_1^\infty 1/x dx = \lim_{t\to\infty} \int_1^t 1/x dx = \lim_{t\to\infty} [ln(x)]_1^t = \lim_{t\to\infty} ln(t) - 0 = \infty.$$

3. Find the average value of $f(x) = \frac{x}{x-6}$ for $0 \leq x \leq 2$.

Solution 1: Making the substitution $u = x - 6 \Rightarrow x = u + 6, du = dx$ we have

$$f_{avg} = \frac{1}{2-0}\int_0^2 \frac{x}{x-6}dx = \frac{1}{2}\int_{-6}^{-4} \frac{u+6}{u}du = \frac{1}{2}\int_{-6}^{-4} 1 + \frac{6}{u}du = \frac{1}{2}\left[u + 6ln|u|\right]_{-6}^{-4}$$

$$= \frac{1}{2}\left[x - 6 + 6ln|x-6|\right]_0^2 = \frac{1}{2}\left[(2 - 6 + 6ln(4)) - (0 - 6 + 6ln(6))\right] = \frac{1}{2}\left[2 + 6ln(4) - 6ln(6)\right].$$

Solution 2: Wouldn't it be nice if the numerator and denominator were the same?

$$f_{avg} = \frac{1}{2-0}\int_0^2 \frac{x}{x-6}dx = \frac{1}{2}\int_0^2 \frac{x-6+6}{x-6}dx = \frac{1}{2}\int_0^2 1dx + \frac{1}{2}\int_0^2 \frac{6}{x-6}dx = \frac{1}{2}\left[x + 6ln|x-6|\right]_0^2$$
$$= \frac{1}{2}\left[(2+6ln(4)) - (0+6ln(6))\right] = \frac{1}{2}\left[(2+6ln(4)) - 6ln(6)\right].$$

4. Find the volume of the solid obtained by rotating the region bounded by the curves $y = 2x$ and $y = x^2$ about the x-axis.

Solution: The curves intersect at $x = 0$. We find the other intersection point: $2x = x^2 \Rightarrow x = 2$. Graph them or note that $2x \geq x^2$ on $[0, 2]$. Washers seem the best choice as then we'll integrate with respect to x and we already have y in terms of x:

$$\pi\int_0^2 r_o^2 - r_i^2 dx = \pi\int_0^2 (2x)^2 - (x^2)^2 dx = \pi\int_0^2 4x^2 - x^4 dx = \pi\left[\frac{4x^3}{3} - \frac{x^5}{5}\right]_0^2 = \pi\left(\frac{32}{3} - \frac{32}{5}\right).$$

5. Let $f(x) = x^2$. Approximate $\int_0^4 x^2\ dx$ using a right hand Riemann sum and 4 subintervals.

Solution: We approximate the area by adding up the area of 4 rectangles, whose width is $\Delta x = \frac{b-a}{n} = \frac{4-0}{4} = 1$, and whose height is determined by the value of $f(x)$ at the right endpoints of each rectangle, $x = 1, 2, 3$, and 4. This yields

$$\int_0^4 x^2 dx \approx \Delta x[f(1) + f(2) + f(3) + f(4)] = (1)[1^2 + 2^2 + 3^2 + 4^2] = 30.$$

6. A honeybee population starts with 100 bees and increases at a rate of $n'(t)$ bees per week. What does $100 + \int_0^4 n'(t)\ dt$ represent?

Solution: By FTC2/Net Change theorem,

$$100 + \int_0^4 n'(t)dt = 100 + n(4) - n(0) = 100 + n(4) - 100 = n(4),$$

and $n(4)$ is just the bee population after 4 weeks.

7. Find the work required to lift a chain lying on the ground to a height of 10 feet if the chain weighs 20 pounds and is 10 feet long.

Solution: Method 1: Chop up the object. We chop up the chain into pieces of length Δx ft. Each piece weighs Δx ft $\times$ 2 lb/ft $= 2\Delta x$ lbs. A piece x ft above ground needs to be lifted up another $10 - x$ ft, so the work required for each piece of chain is $2(10-x)\Delta x$. Adding up the work required for each piece gives $\int_0^{10} 2(10-x)dx = 100J$. **Solution:** Method 2: Chop up the distance. Let x be the height above the ground. We'll add up the work required to lift the chain each tiny distance Δx. If we're at a height of x feet, we have to lift x feet of chain and the rest is coiled up on the ground. The force to move each tiny distance $= x$ feet $\times$ 2 lb/ft $= 2x$ lb, so the total work is $W = \int_0^{10} 2x\ dx = 100J$.