Math 3B Final Review
New material

u-substitution: works for integrating compositions of functions; pick u to be the 'inside' function.

Integration by parts - undoing the product rule: $\int u \ \mathrm{d}v = uv - \int v \ \mathrm{d}u$.
Generally, picking u in this descending order works, and $\mathrm{d}v$ is what's left:
Inverse trig
Logarithm
Algebraic (polynomial)
Trig
Exponential

Trig substitutions and integrals: See separate handout.

Partial fractions: -
If necessary, make a substitution to get a ratio of polynomials
If the degree of the numerator is $\geq$ the degree of denominator, do long division first.
Then factor the denominator into linear terms and irreducible quadratics.

factor in denominator		term in partial fraction decomposition
$(ax+b)^k$	$\Rightarrow$	$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + ... + \frac{A_k}{(ax+b)^k}$
$(ax^2+bx+c)^k$	$\Rightarrow$	$\frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + ... + \frac{A_kx+B_k}{(ax^2+bx+c)^k}$

Misc: Sometimes you'll need to "complete the square": eg: $x^2+6x+5 = x^2+6x+9-9+5 = (x+3)^2-4$ (divide x coefficient by 2, square it, and add and subtract it. Note: works when coefficient of x^2 is 1)

Improper integrals

Type 1: infinite interval: $\int_a^\infty f(x)dx = \lim_{t\to\infty} \int_a^t f(x)dx$, $\int_{-\infty}^b f(x)dx = \lim_{t\to-\infty} \int_t^b f(x)dx$

Type 2: discontinuity in interval: -

f discontinuous at a: $\int_a^b f(x)dx = \lim_{t\to a^+} \int_t^b f(x)dx$

f discontinuous at b: $\int_a^b f(x)dx = \lim_{t\to b^-} \int_a^t f(x)dx$

f discontinuous at $c, a < c < b$: $\int_a^b f(x)dx = \lim_{t\to c^-} \int_a^t f(x)dx + \lim_{t\to c^+} \int_t^b f(x)dx$

Note!: It is possible that an integral is both Type 1 and Type 2.

Old material

Definite integral: Suppose $f(x)$ is continuous on $[a,b]$. Divide $[a,b]$ into subintervals of length $\Delta x = \frac{b-a}{n}$ and choose a sample point x_i^* from each subinterval. Then

$$\int_a^b f(x) \ \mathrm{d}x = \lim_{n\to\infty} \sum_{i=1}^n f(x_i^*)\Delta x = \lim_{n\to\infty} \Delta x \sum_{i=1}^n f(x_i^*).$$

Approximating integrals with Riemann sums: Choose n = number of rectangles and choose a sample point x_i^* (usually left, right, or mid) from each subinterval. Then

$$\int_a^b f(x) \ \mathrm{d}x \approx \sum_{i=1}^n f(x_i^*)\Delta x = \Delta x\big[f(x_1^*) + f(x_2^*) + ... + f(x_n^*)\big] = A_n, \text{ where } \Delta x = \tfrac{b-a}{n}.$$

Each A_n is called a **Riemann sum**. If $x_i^* =$ right endpoint, then $x_i^* = a + i \cdot \Delta x$.

For an increasing function, using left endpoints gives a lower bound and using right endpoints gives an upper bound.
For a decreasing function, using right endpoints gives a lower bound and using left endpoints gives an upper bound.

Antiderivative: An anti-derivative of $f(x)$ is a function $F(x)$ such that $F' = f$.

Indefinite integral: $\int f(x)\ \mathrm{d}x = F(x) + C$, where F is an anti-derivative of f.

FTC ("integration and differentiation are inverse processes")

Part 1: $\frac{d}{dx} \int_a^x f(t)\ \mathrm{d}t = f(x)$.

Part 1 w/ Chain Rule: $\frac{d}{dx} \int_a^{g(x)} f(t)\ \mathrm{d}t = f(g(x)) \cdot g'(x)$.

Part 2: $\int_a^b F'(x)\ \mathrm{d}x = F(b) - F(a)$
Main application of FTC2: the integral of the rate of change of $F(x)$ is the net change in $F(x)$ from $x = a$ to $x = b$.
eg, if $v(t) =$ velocity, then $\int_{t_1}^{t_2} v(t)\ \mathrm{d}t =$ net distance traveled $=$ net change in position from time t_1 to $t_2 =$ displacement (*not* total distance traveled (in general)).

Applications

Area between curves: First find where the curves intersect. Then do $\int_a^b$[top function] $-$ [bottom function] $\mathrm{d}x$ or $\int_c^d$[right function] $-$ [left function] $\mathrm{d}y$

Average value of a function $f(x)$ between $x = a$ and $x = b$: $\frac{1}{b-a} \int_a^b f(x)\ \mathrm{d}x$

Volume: We can find the volume of a solid by adding up areas of cross sections of the solid. The main formula is $\int_a^b A(x)\ \mathrm{d}x$ or $\int_c^d A(y)\ \mathrm{d}y$ where $A(x), A(y)$ give the area of a cross section of the solid. The two main cases are:

Disks/Washers: $A = \pi\big((\text{outer radius})^2 - (\text{inner radius})^2\big)$. Cross sections are perpendicular to the axis of rotation.

Cylindrical she||s: $A = 2\pi(\text{radius})(\text{height})$. Cross sections are parallel to the axis of rotation.

Work = Force × Distance:

Method I: Distance in pieces: Chop up the distance and add up the work required to move each tiny distance $\Delta x \Rightarrow W = \int_a^b \text{force}\ dx$.

Method II: Object in pieces: Chop up the object and add up the work required to move each piece the *whole* distance $\Rightarrow W = \int_a^b \text{force} \times \text{distance}\ dx$.

Hooke's Law: Force required to stretch a spring x units beyond natural length is proportional to $x : f(x) = kx$.

Useful formulas: Force = mass × acceleration, mass = density · volume

Note: Pounds = unit of force, Kg = unit of mass. $g = 9.8\ \text{m/s}^2 = 32\ \text{ft}/s^2$.

Units: kg-m-sec ⇒ Joules, lb-ft-sec ⇒ ft-lbs

More applications (not on the final)

Arc length

$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$ if $y = f(x), a \leq x \leq b$.

$L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy$ if $x = g(y), c \leq y \leq d$.

Arc length function: $s(x) = \int_a^x \sqrt{1 + [f'(t)]^2} dt$ = length of arc from the point $(a, f(a))$ to $(x, f(x))$.

Surface area of a solid of revolution

Rotation about x-axis: $S = 2\pi \int y \ \mathrm{d}s$,

Rotation about y-axis: $S = 2\pi \int x \ \mathrm{d}s$,

where $ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$ if $y = f(x), a \leq x \leq b$.

$ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy$ if $x = g(y), c \leq y \leq d$.

Center of mass

Let ρ be the uniform density of a plate that is the region bounded by the curves $f(x)$ and $g(x)$, where $f(x) \geq g(x)$.

Moments M_x and M_y: measure the tendency of a region to rotate about the x- and y-axis, respectively:
$M_x = \rho \int_a^b \frac{1}{2} \left([f(x)]^2 - [g(x)]^2\right) dx$, $M_y = \rho \int_a^b x(f(x) - g(x)) dx$.

Center of mass: Let $A = \int_a^b f(x) - g(x) dx$ be the area of the plate and $M = \rho \times A$ be the mass of the plate. Then the coordinates of the center of mass $(\overline{x}, \overline{y})$ are:

$$\overline{x} = \frac{M_y}{M} = \frac{\int_a^b x(f(x) - g(x)) dx}{A}, \text{ and } \overline{y} = \frac{M_x}{M} = \frac{\int_a^b \frac{1}{2} \left([f(x)]^2 - [g(x)]^2\right) dx}{A}$$

Hydrostatic Force

Hydrostatic Pressure: $P = \rho g d$, where ρ = (mass) density of fluid, $g = 9.8$ m/s^2, d = depth below surface.

Hydrostatic Force: $F = \int_a^b P \times A \ \mathrm{d}x$, where A is the area of strips of height Δx and width determined by the function.