# Math 4A Final practice problems

---

1. Describe all least-squares solutions to $A\mathbf{x} = \mathbf{b}$ where $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 3 \\ 8 \\ 2 \end{bmatrix}$.

   What is the distance between $\mathbf{b}$ and any $A\mathbf{x}^*$ (this is called the least-squares error)?

2. Show that $B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthogonal basis of $\mathbb{R}^3$ and express $\mathbf{x}$ as a linear combination of the $\mathbf{u}$'s, find the coordinates of $\mathbf{x}$ relative to $B$, and find the coordinate vector $[\mathbf{x}]_B$ of $\mathbf{x}$ relative to $B$.

   $$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} 8 \\ -4 \\ 3 \end{bmatrix}.$$

   Then normalize the $\mathbf{u}$'s to produce an orthonormal basis.

3. Find a basis for the orthogonal complement $\mathbf{u}_\perp$ to $\mathbf{u} = (0, 1, 0, 2)$. Is $(1, -4, 3, 2)$ in $\mathbf{u}_\perp$?

4. Find a unit vector in the opposite direction of $\begin{bmatrix} -3 \\ 4 \end{bmatrix}$.

5. Find a nonzero vector perpendicular to both $\begin{bmatrix} -3 \\ -8 \\ -2 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}$.

6. Let $A = \begin{bmatrix} 4 & -3 \\ 2 & -1 \end{bmatrix}$. Compute $A^8$.

7. Diagonalize $A = \begin{bmatrix} 2 & 4 & 3 \\ -4 & -6 & 3 \\ 3 & 3 & 1 \end{bmatrix}$ or justify why this is not possible.

8. Let $A = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 1 & 4 & -3 & 0 \\ -1 & -2 & 0 & -3 \end{bmatrix}$.

   Find a basis for which the linear transformation given by $A$ is a diagonal matrix.

9. The matrix $A = \begin{bmatrix} -5 & 0 & 8 \\ 4 & -1 & -8 \\ -2 & 0 & 3 \end{bmatrix}$ has one real eigenvalue.

   Find this eigenvalue, its multiplicity, and a basis of the corresponding eigenspace. What is the dimension of the eigenspace? Is $A$ diagonalizable? Why or why not?

10. The matrix $A = \begin{bmatrix} 2 & 1 \\ 0 & k \end{bmatrix}$ has exactly one real eigenvalue if and only if $k$ is what number?

    Find the dimension of the eigenspace in this case.

11. Find a basis for the subspace of $\mathbb{R}^3$ spanned by $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$, where

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 3 \\ 1 \\ -4 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}, \mathbf{v}_4 = \begin{bmatrix} -1 \\ -1 \\ -2 \end{bmatrix}.$$

12. If $A$ is a $7 \times 9$ matrix with a two-dimensional null space, what is the rank of $A$? Is $T(\mathbf{x}) = A\mathbf{x}$ onto? One-to-one?

13. If $T : \mathbb{R}^9 \to \mathbb{R}^6$ is a linear transformation, could $T$ have a 7-dimensional kernel?

14. Let $A$ be an $n \times n$ matrix. Suppose you find that the dimension of the null space of $A$ is 1 - is $A$ invertible in this case? Is $T(\mathbf{x}) = A\mathbf{x}$ one-to-one? Suppose you find the dimension of the column space of $A$ (the rank of $A$) is $n$ - is $A$ invertible in this case? Is $T(\mathbf{x}) = A\mathbf{x}$ onto?

15. Suppose you find that the eigenvalues of a $3 \times 3$ matrix $A$ are 0, 1, and 2 - is $A$ invertible?

16. Suppose you compute the determinant of a $4 \times 4$ matrix $A$ to be 24. Do the columns of $A$ form a basis for $\mathbb{R}^4$?

17. Let $A$ be a $4 \times 4$ matrix with rows $v_1, v_2, v_3, v_4$ and suppose $\det A = -2$.

    Compute $\det \begin{bmatrix} v_4 \\ 5v_2 + 2v_3 \\ 6v_2 + 7v_3 \\ v_1 \end{bmatrix}$.

**Selected webwork problems** (I'm referring to what the webworks are called on Gauchospace, which doesn't always match up with webwork):

1-1: 5
1-2: 1, 2, 10
2-1: 11
3-1: 11 3-2: 5 4-1: 8
4-2: 5
5-1: 1, 6, 10, 12
5-2: 7, 8
6-1: 1, 3, 5, 8
6-2: 2
7-1: 6, 7
7-2: 1, 2, 5, 7, 8
8-1: 7, 9, 12
8-2: 7
9-1: 12
9-2: 3, 8