1. We need to solve $A^T A \vec{x} = A^T \vec{b}$

$$\downarrow$$

$$\begin{bmatrix} 4 & 2 & 2 \\ 2 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix} \vec{x} = \begin{bmatrix} 14 \\ 4 \\ 10 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 4 & 2 & 2 & 14 \\ 2 & 2 & 0 & 4 \\ 2 & 0 & 2 & 10 \end{array}\right] \text{ has RREF } \left[\begin{array}{ccc|c} 1 & 0 & 1 & 5 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \begin{array}{l} x_1 = 5 - s \\ x_2 = -3 + s \\ x_3 = s \end{array}$$

All least-squares solutions $\hat{x}$ are of the form $\hat{x} = \begin{bmatrix} 5 \\ -3 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$.

Taking $s = 0$, we have $x^* = \begin{bmatrix} 5 \\ -3 \\ 0 \end{bmatrix}$

and $Ax^* = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ -3 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 5 \\ 5 \end{bmatrix}$ and $\| \vec{b} - A\vec{x}^* \| = \sqrt{(1-2)^2 + (3-2)^2 + (8-5)^2 + (2-5)^2}$

$$= \sqrt{20}.$$

2. $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ is an orthogonal set since $\vec{u}_1 \cdot \vec{u}_2 = \vec{u}_1 \cdot \vec{u}_3 = \vec{u}_2 \cdot \vec{u}_3 = 0$

One way to show they form a basis for $\mathbb{R}^3$ is to cite the facts that orthogonal sets are L.I., and 3 linearly independent vectors in $\mathbb{R}^3$ must span $\mathbb{R}^3$ (or show that $[\vec{u}_1 \ \vec{u}_2 \ \vec{u}_3]$ has 3 pivots).

The other parts of ~~#~~2 are all the same.

Want to write $\vec{x} = c_1 \vec{u}_1 + c_2 \vec{u}_2 + c_3 \vec{u}_3$. Use $c_j = \dfrac{\vec{x} \cdot \vec{u}_j}{\vec{u}_j \cdot \vec{u}_j} = \dfrac{\vec{x} \cdot \vec{u}_j}{\|\vec{u}_j\|^2}$

$\vec{x} \cdot \vec{u}_1 = 11$, $\vec{u}_1 \cdot \vec{u}_1 = 2$, $\quad \vec{x} \cdot \vec{u}_2 = -21$, $\vec{u}_2 \cdot \vec{u}_2 = 18$, $\quad \vec{x} \cdot \vec{u}_3 = 6$, $\vec{u}_3 \cdot \vec{u}_3 = 9$

so $\vec{x} = \frac{11}{2} \vec{u}_1 - \frac{21}{18} \vec{u}_2 + \frac{6}{9} \vec{u}_3$.

The c_j's are the coordinates of $\vec{x}$ relative to $\mathcal{B}$,

and $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} \frac{11}{2} \\ \frac{-21}{18} \\ \frac{6}{9} \end{bmatrix}$.

To produce an orthonormal basis, divide each $\vec{u}_j$ by its norm:

$$\left\{ \frac{\vec{u}_1}{\sqrt{2}}, \frac{\vec{u}_2}{\sqrt{18}}, \frac{\vec{u}_3}{3} \right\}.$$

3. $(0,1,0,2)\cdot(x,y,z,w) = y+2w=0 \rightarrow [0\ 1\ 0\ 2 \mid 0]$

$x = r$
$y = -2t$ $\rightarrow$ Basis is $\left\{ \begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\-2\\0\\1\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\end{bmatrix} \right\}$.
$z = s$
$w = t$

$(1,-4,3,2)$ is in $\vec{u}_\perp$ since $(0,1,0,2)\cdot(1,-4,3,2) = 0$.

4. The norm of $\begin{bmatrix}-3\\4\end{bmatrix}$ is $\sqrt{(-3)^2+4^2} = 5$.
The vector we want is then $-\frac{1}{5}\begin{bmatrix}-3\\4\end{bmatrix}$.

5. $\begin{bmatrix}-3\\-8\\-2\end{bmatrix}\cdot\begin{bmatrix}x\\y\\z\end{bmatrix} = -3x-8y-2z=0$

$\begin{bmatrix}1\\3\\-1\end{bmatrix}\cdot\begin{bmatrix}x\\y\\z\end{bmatrix} = x+3y-z=0$

$\begin{bmatrix}-3 & -8 & -2 & \mid & 0\\ 1 & 3 & -1 & \mid & 0\end{bmatrix}$ has RREF $\begin{bmatrix}1 & 0 & 14 & \mid & 0\\ 0 & 1 & -5 & \mid & 0\end{bmatrix}$ $\begin{matrix}x=-14s\\ y=5s\\ z=s\end{matrix}$

Set $s=1$ to get that $\begin{bmatrix}-14\\5\\1\end{bmatrix}$ is perpendicular to both vectors.

6. Start by diagonalizing A.

$$\begin{vmatrix}4-\lambda & -3\\ 2 & -1-\lambda\end{vmatrix} = (\lambda-1)(\lambda-2)=0$$

$\lambda_1=1$: $\begin{bmatrix}3 & -3 & \mid & 0\\ 2 & -2 & \mid & 0\end{bmatrix} \rightarrow \begin{bmatrix}1 & -1 & \mid & 0\\ 0 & 0 & \mid & 0\end{bmatrix} \rightarrow v_1 = \begin{bmatrix}1\\1\end{bmatrix}$ $\rightarrow P = \begin{bmatrix}1 & 3\\ 1 & 2\end{bmatrix}$, $P^{-1} = \begin{bmatrix}-2 & 3\\ 1 & -1\end{bmatrix}$

$\lambda_2=2$: $\begin{bmatrix}2 & -3 & \mid & 0\\ 2 & -3 & \mid & 0\end{bmatrix} \rightarrow \begin{bmatrix}1 & -\frac{3}{2} & \mid & 0\\ 0 & 0 & \mid & 0\end{bmatrix} \rightarrow v_2 = \begin{bmatrix}3\\2\end{bmatrix}$ $D = \begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix}$.

$$A^8 = PD^8P^{-1} = \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1^8 & 0 \\ 0 & 2^8 \end{bmatrix}\begin{bmatrix} -2 & 3 \\ 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 3\cdot 2^8 \\ 1 & 2^9 \end{bmatrix}\begin{bmatrix} -2 & 3 \\ 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -2+3\cdot 2^8 & 3-3\cdot 2^8 \\ -2+2^9 & 3-2^9 \end{bmatrix}.$$

7. $$\begin{vmatrix} 2-\lambda & 4 & 3 \\ -4 & -6-\lambda & 3 \\ 3 & 3 & 1-\lambda \end{vmatrix} = (2-\lambda)\left[(-6-\lambda)(1-\lambda)-9\right] - 4\left[-4(1-\lambda)-9\right] + 3\left[-12-3(-6-\lambda)\right]$$

$$= (2-\lambda)(\lambda^2+5\lambda-15) + 52 - 16\lambda + 18 + 9\lambda$$

$$= -\lambda^3 - 3\lambda^2 - 5\lambda - 30 + 70 - 7\lambda$$

$= -\lambda^3 - 3\lambda^2 - 12\lambda + 40 = 0$. Possible roots are $\dfrac{\pm 1, 2, 5, 8, 10, 20, 40}{\pm 1}$

Check that $\lambda_1 = 4$, $\lambda_2 = -2$, $\lambda_3 = -5$ are the roots so

$-\lambda^3 - 3\lambda^2 - 12\lambda + 40 = -(\lambda-4)(\lambda+2)(\lambda+5)$.

$$\lambda_1 = 4: \quad \left[\begin{array}{ccc|c} -2 & 4 & 3 & 0 \\ -4 & -10 & 3 & 0 \\ 3 & 3 & -3 & 0 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & -\frac{7}{6} & 0 \\ 0 & 1 & \frac{1}{6} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \to v_1 = \begin{bmatrix} 7 \\ -1 \\ 6 \end{bmatrix}.$$

$$\lambda_2 = -2: \quad \left[\begin{array}{ccc|c} 4 & 4 & 3 & 0 \\ -4 & -4 & 3 & 0 \\ 3 & 3 & 3 & 0 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \to v_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}.$$

$$\lambda_3 = -5: \quad \left[\begin{array}{ccc|c} 7 & 4 & 3 & 0 \\ -4 & -1 & 3 & 0 \\ 3 & 3 & 6 & 0 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & -\frac{5}{3} & 0 \\ 0 & 1 & \frac{11}{3} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \to v_3 = \begin{bmatrix} 5 \\ -11 \\ 3 \end{bmatrix}.$$

So A is diagonalizable and $A = PDP^{-1}$ where $P = \begin{bmatrix} 7 & -1 & 5 \\ -1 & 1 & -11 \\ 6 & 0 & 3 \end{bmatrix}$ and $D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -5 \end{bmatrix}$.

8. A basis of eigenvectors is what we're after.

Since A is lower triangular, it's eigenvalues are its diagonal entries, $\lambda_1 = 5$, $\lambda_2 = -3$.

$$\lambda_1 = 5: \left[\begin{array}{cccc|c} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 4 & -8 & 0 & 0 \\ -1 & -2 & 0 & -8 & 0 \end{array}\right] \rightsquigarrow \left[\begin{array}{cccc|c} 1 & 0 & 8 & 16 & 0 \\ 0 & 1 & -4 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \vec{v}_1 = \begin{bmatrix} -8 \\ 4 \\ 1 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} -16 \\ 4 \\ 0 \\ 1 \end{bmatrix}.$$

$$\lambda_2 = -3: \left[\begin{array}{cccc|c} 8 & 0 & 0 & 0 & 0 \\ 0 & 8 & 0 & 0 & 0 \\ 1 & 4 & 0 & 0 & 0 \\ -1 & -2 & 0 & 0 & 0 \end{array}\right] \rightsquigarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \vec{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \vec{v}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Then $B = \{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ is a basis for which the linear transformation given by A is a diagonal matrix.

(Why? With $P = [\vec{v}_1\ \vec{v}_2\ \vec{v}_3\ \vec{v}_4]$ and $D = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix}$,

$A = PDP^{-1} \longrightarrow D = P^{-1}AP$.

P is the change of coordinate matrix from B to S

P^{-1} " " " " " " " " S to B.

so $D_B = P^{-1}AP_B$ just says: P converts $[x]_B$ to standard basis S,

then multiply by A,

then P^{-1} converts back to the basis B,

so D does what A does, just in the basis B instead of S).

9. $\begin{vmatrix} -5-\lambda & 0 & 8 \\ 4 & -1-\lambda & -8 \\ -2 & 0 & 3-\lambda \end{vmatrix}$ $= (-1-\lambda)[(-5-\lambda)(3-\lambda)+16]$

$= (-1-\lambda)(\lambda^2+2\lambda+1) = (-1-\lambda)(\lambda-1)^2 = -(\lambda-1)^3 = 0$

$\to \lambda = 1$

$\lambda=1$ has multiplicity 3.

$\left[\begin{array}{ccc|c} -4 & 0 & 8 & 0 \\ 4 & 0 & -8 & 0 \\ -2 & 0 & 4 & 0 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \longrightarrow$ Basis for eigenspace E_{-1} is

$\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\}$

Since we don't have 3 L. I. eigenvectors A is not diagonalizable.

10. Solution 1: A is upper triangular, so its eigenvalues are its diagonal entries, and thus $K=2$.

Solution 2: $\begin{vmatrix} 2-\lambda & 1 \\ 0 & K-\lambda \end{vmatrix} = (2-\lambda)(K-\lambda) = \lambda^2 - (2+K)\lambda + 2K = 0$

$\lambda = \dfrac{2+K \pm \sqrt{(2+K)^2 - 8K}}{2}$. Exactly one real eigenvalue

$\leftrightarrow (2+K)^2 - 8K = 0$

$\leftrightarrow K^2 - 4K + 4 = 0$

$\leftrightarrow (K-2)^2 = 0$

$\leftrightarrow K = 2$.

$\left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \longrightarrow$ Basis of eigenspace is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, so E_2 is 1-dimensional.

11. (note that for $A = [\vec{v}_1\ \vec{v}_2\ \vec{v}_3\ \vec{v}_4]$, this the same as asking for a basis for $\text{Col } A = \text{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$)

Row reduce A to get its RREF to be $\begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$
(can stop at REF)

$\vec{v}_1$ and $\vec{v}_2$ form a basis for $\text{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ since they are the pivot columns of A.

(The next few questions relate to the rank-nullity theorem which has a few forms:

If A is an $m \times n$ matrix, $\dim \text{Nul } A + \dim \text{Col } A = n$
$\text{nullity } A + \text{rank } A = n$

If $T: V \to W$ is a lin. trans, $\dim \text{Ker } T + \dim \text{Im } T = \dim V$
$\text{nullity } T + \text{rank } T = \dim V$

To remember: # nonpivot columns + # pivot columns = # columns)

12. We have $2 + \text{rank } A = 9$, so $\text{rank } A = 7$.
T is onto $\longleftrightarrow$ $\text{rank } A = 7$ so
and T is 1-1 $\longleftrightarrow$ $\text{nullity } A = 0$
T is one-to-one but not onto $\mathbb{R}^7$.

13. We have $\dim \text{Ker } T + \dim \text{Im } T = 9$
Since $7 + \dim \text{Im } T = 9 \to \dim \text{Im } T = 2$

This is possible, so T can have a 7-dimensional kernel. For example $T(x) = Ax$ where
$A = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$

14. A is invertible $\longleftrightarrow$ $\dim \text{nul } A = 0$ $\longleftrightarrow$ T is one-to-one, so no.
A is invertible $\longleftrightarrow$ $\dim \text{col } A = n$ $\longleftrightarrow$ T is onto, so yes.

15. A is invertible if and only if 0 is not an eigenvalue, so <u>no</u>
(Why? 0 is an eigenvalue $\longleftrightarrow$ $A\vec{v} = 0\vec{v} = \vec{0}$ for some nonzero $\vec{v}$ $\longleftrightarrow$ $\text{nul } A \neq \{\vec{0}\}$
$\longleftrightarrow$ A is not invertible)

16. A is invertible $\longleftrightarrow$ The columns of A form a basis for $\mathbb{R}^4$
$\longleftrightarrow$ $\det A \neq 0$, so <u>yes</u>.

17. Solution 1: This will work for any A with $\det A = -2$, so let $A = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

Then $\begin{vmatrix} v_4 \\ 5v_2+2v_3 \\ 6v_2+7v_3 \\ v_1 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 5 & 2 & 0 \\ 0 & 6 & 7 & 0 \\ -2 & 0 & 0 & 0 \end{vmatrix} = -(-2)\begin{vmatrix} 0 & 0 & 1 \\ 5 & 2 & 0 \\ 6 & 7 & 0 \end{vmatrix} = 2(1)\begin{vmatrix} 5 & 2 \\ 6 & 7 \end{vmatrix}$

$= 2(23)$

$= 46$.

Solution 2: (I avoid using the row operation $R_i \to cR_i$)

$$\begin{vmatrix} v_4 \\ 5v_2+2v_3 \\ 6v_2+7v_3 \\ v_1 \end{vmatrix} = -\begin{vmatrix} v_1 \\ 5v_2+2v_3 \\ 6v_2+7v_3 \\ v_4 \end{vmatrix} \overset{R_2 - \frac{2}{7}R_3}{=} -\begin{vmatrix} v_1 \\ \frac{23}{7}v_2 \\ 6v_2+7v_3 \\ v_4 \end{vmatrix}$$

$$= -\frac{23}{7}\begin{vmatrix} v_1 \\ v_2 \\ 6v_2+7v_3 \\ v_4 \end{vmatrix}$$

$$\overset{R_3 - 6R_2}{=} -\frac{23}{7}\begin{vmatrix} v_1 \\ v_2 \\ 7v_3 \\ v_4 \end{vmatrix}$$

$$= -\frac{23}{7}\cdot 7\begin{vmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{vmatrix} = -23 \det A = -23(-2)$$

$= 46$

Solution 3: (using $R_i \to cR_i$)

$$\begin{vmatrix} v_4 \\ 5v_2+2v_3 \\ 6v_2+7v_3 \\ v_1 \end{vmatrix} = -\begin{vmatrix} v_1 \\ 5v_2+2v_3 \\ 6v_2+7v_3 \\ v_4 \end{vmatrix} \overset{7R_2}{=} -\frac{1}{7}\begin{vmatrix} v_1 \\ 35v_2+14v_3 \\ 6v_2+7v_3 \\ v_4 \end{vmatrix}$$

$$\overset{R_2-2R_3}{=} -\frac{1}{7}\begin{vmatrix} v_1 \\ 23v_2 \\ 6v_2+7v_3 \\ v_4 \end{vmatrix}$$

$$= \frac{-23}{7}\begin{vmatrix} v_1 \\ v_2 \\ 6v_2+7v_3 \\ v_4 \end{vmatrix}$$

$$\overset{R_3-6R_2}{=} \frac{-23}{7}\begin{vmatrix} v_1 \\ v_2 \\ 7v_3 \\ v_4 \end{vmatrix}$$

$$= \frac{-23}{7}\cdot 7\begin{vmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{vmatrix}$$

$= -23 \det A$

$= -23(-2)$

$= 46$