Post-Midterm 2

A linear first order DE system can be written as $\mathbf{x}'(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{f}(t)$. The system is **homogeneous** if $\mathbf{f}(t) = \mathbf{0}$.

Existence and Uniqueness for Linear DE Systems: The IVP $\mathbf{x}'(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{f}(t), \mathbf{x}(t_0) = \mathbf{x}_0$, has a unique solution on an open interval I containing t_0 if $\mathbf{A}(t)$ and $\mathbf{f}(t)$ are both continuous on I.

The general solution to a homogeneous system of n first order linear DEs is $c_1\mathbf{x}_1(t) + ... + c_n\mathbf{x}_n(t)$, where $\mathbf{x}_1(t), ..., \mathbf{x}_n(t)$ are n LI solutions. We say that $\mathbf{x}_1(t), ..., \mathbf{x}_n(t)$ form a **fundamental set** of solutions.

The general solution to a non-homogeneous linear first order DE system is $\mathbf{x} = \mathbf{x}_h + \mathbf{x}_p$, where $\mathbf{x}_h$ is the general solution to the associated homogeneous system, and $\mathbf{x}_p$ is any particular solution to the non-homogeneous system.

Solving homogeneous linear 2×2 DE systems with constant coefficients, $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

- Case 1: (Possibly equal) real eigenvalues λ_1, λ_2, LI eigenvectors $\mathbf{v}_1, \mathbf{v}_2$.

  General solution: $\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2$.

- Case 2: Complex conjugate eigenvalues $\lambda_1, \lambda_2 = a \pm bi$, complex conjugate eigenvectors $\mathbf{v}_1, \mathbf{v}_2 = \mathbf{p} \pm i\mathbf{q}$.

  General solution: $\mathbf{x}(t) = c_1 e^{at}(\cos(bt)\mathbf{p} - \sin(bt)\mathbf{q}) + c_2 e^{at}(\sin(bt)\mathbf{p} + \cos(bt)\mathbf{q})$.

- Case 3: Repeated real eigenvalue λ, one LI eigenvector $\mathbf{v}$. Solve $(\mathbf{A} - \lambda\mathbf{I})\mathbf{u} = \mathbf{v}$ to find a nonzero vector $\mathbf{u}$, called a **generalized eigenvector** of $\mathbf{A}$ corresponding to λ.

  General solution: $\mathbf{x}(t) = c_1 e^{\lambda t}\mathbf{v} + c_2 e^{\lambda t}(t\mathbf{v} + \mathbf{u})$.

Note that this generalizes to $n \times n$ systems. The above contribute terms to the general solution.

Summary of phase portraits: see page 2 of PDF on my course page. The borderline cases on page 3 are not as important.

For $\mathbf{x}' = A\mathbf{x}$, $\mathbf{0}$ is a source (unstable) if nearby solutions move away, a sink (asymptotically stable) if nearby solutions move towards, and a saddle (unstable) if nearby solutions exhibit both behaviors. In the case of purely imaginary eigenvalues, $\mathbf{0}$ is neutrally stable. If $|\lambda_1| > |\lambda_2|$, $\mathbf{v}_1$ is called the fast eigenvector $\mathbf{v}_1$ is called the fast eigenvector. $c_1 e^{\lambda_1 t}$ approaches $\pm\infty$ or 0 (depending on sign) fastest, affecting solution trajectories as $t \to \infty$.

An $n \times n$ matrix A is diagonalizable if and only if A has n LI eigenvectors. In this case, $A = PDP^{-1}$, where D is a diagonal matrix of eigenvalues and P is a matrix are eigenvectors listed in the same order as the eigenvalues of D.

The change of variables $\mathbf{x} = \mathbf{P}\mathbf{w}$ transforms a homogeneous linear DE system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ (with diagonalizable matrix $\mathbf{A}$) into a decoupled system $\mathbf{w}' = \mathbf{D}\mathbf{w}$.

The change of variables $\mathbf{x} = \mathbf{P}\mathbf{w}$ transforms a non-homogeneous linear DE system $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{f}(t)$ (with diagonalizable matrix $\mathbf{A}$) into a decoupled system $\mathbf{w}' = \mathbf{D}\mathbf{w} + \mathbf{P}^{-1}\mathbf{f}(t)$. The decoupled system can be solved using methods for first order DEs. Then use $\mathbf{x} = \mathbf{P}\mathbf{w}$ to find $\mathbf{x}$.

For an (autonomous) non-linear system $x' = f(x, y), y' = g(x, y)$, equilibrium solutions occur where $x' = y' = 0$. If (a, b) is an equilibrium point, make the substitution $u = x - a, v = y - b$ to translate the equilibrium point to $(0, 0)$. By writing x' and y' in terms of u and v and dropping non-linear terms (which are small since u and v are close to 0), we see that near the equilibrium point, the system behaves like the linear system $\mathbf{u}' = A\mathbf{u}$. Thus we can describe the nature of the equilibrium point and nearby solutions using our previous methods.

Post-Midterm 1, Pre-Midterm 2

The most general 2nd order linear DE is $p(t)y'' + q(t)y' + r(t)y = g(t)$. If $g(t) = 0$, the DE is **homogeneous**.
For the DE $ay'' + by' + cy = 0$ we assume $y(t) = e^{rt}$. This leads to the **characteristic equation** $ar^2 + br + c = 0$.

- Case 1: Distinct real roots r_1, r_2. General solution: $y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$.
- Case 2: Complex roots $r_1, r_2 = a \pm bi$. General solution: $y(t) = e^{at}(c_1 \cos(bt) + c_2 \sin(bt))$.
- Case 3: Repeated real root r. General solution: $y(t) = c_1 e^{rt} + c_2 t e^{rt}$.

Existence and Uniqueness for 2nd order linear DEs: For the IVP

$$y'' + p(t)y' + q(t)y = g(t), \qquad y(t_0) = y_0, \qquad y'(t_0) = y_0',$$

there exists a unique solution to the IVP on the interval $I = (a, b)$ containing t_0 if $p(t), q(t)$ and $g(t)$ are continuous on I.

Solutions to $y'' + p(t)y' + q(t)y = 0$ form a 2-dimensional vector space. What this tells us is that given 2 linearly independent solutions y_1 and y_2, $c_1 y_1 + c_2 y_2$ is also a solution, and every solution is of this form. We say that y_1 and y_2 form a **fundamental set** of solutions.

Wronskian test for linear independence: If $W(f, g) = fg' - f'g \neq 0$, then f and g are linearly independent.

The general solution of the linear nonhomogeneous DE $y'' + p(t)y' + q(t)y = f(t)$ is $y = c_1 y_1 + c_2 y_2 + y_p$, where y_1 and y_2 are linearly independent solutions to the associated homogeneous DE and y_p is any particular solution of the nonhomogeneous DE.

Reduction of order: Given a solution y_1 to a 2nd order linear homogeneous DE $p(t)y'' + q(t)y' + r(t)y = 0$, we assume that y_2 is of the form $y_2(t) = v(t)y_1(t)$. Upon substituting y_2 into our DE, terms involving v will disappear, and we can solve the resulting 1st order DE by making the substitution $w = v', w' = v''$ and using separation of variables or the integrating factor method.

Method of **undetermined coefficients** for solving the 2nd order linear nonhomogeneous DE $y'' + p(t)y' + q(t)y = f(t)$.

1. First find the general solution $y_h = c_1y_1 + c_2y_2$ of the associated homogeneous equation.

2. Guess y_p according to what $f(t)$.

$f(t)$	y_p
polynomial	$A_nt^n + ... + A_0$where n is the degree of f
sin's and cos's	$A\cos(\omega t) + B\sin(\omega t)$
exponential	Ae^{kt}
product/sum of the above	product/sum of the above

3. Compare your guess to y_h and multiply by t (once or twice) if any part of y_p shows up in y_h.

4. Compute y_p', y_p'', substitute in to DE, equate coefficients and solve.

5. The general solution is then $y = y_h + y_p$.

Method of **variation of parameters** for solving the 2nd order linear nonhomogeneous DE $y'' + p(t)y' + q(t)y = g(t)$

1. Find the general solution $y_h = c_1y_1 + c_2y_2$ to the associated homogeneous DE.

2. A particular solution is given by

$$y_p = -y_1 \int \frac{y_2 g}{W(y_1, y_2)} dt + y_2 \int \frac{y_1 g}{W(y_1, y_2)} dt.$$

   We can assume both constants of integration are 0.

3. The general solution is then $y = y_h + y_p$.

Pre-Midterm 1

We've considered **difference equations** of the form $y_{n+1} = ay_n + b$. It's called a difference equation because it can be rewritten as $y_{n+1} - y_n = (a-1)y_n + b$. Its **solution** is $y_n = a^n y_0 + b\left(\frac{1-a^n}{1-a}\right)$ (provided $a \neq 1$). This can be seen by using the fact that $\sum_{k=0}^{n-1} a^k = \frac{1-a^n}{1-a}$.

We can model interest problems with difference equations - think about what y_n, a, and b represent.

The family of solutions to a DE are called **integral curves**.

A **slope field** or **direction field** for a DE $y' = f(t, y)$ is a graph of tick marks in the ty-plane where the tick mark through the point (t, y) has slope $f(t, y)$.

An **isocline** is a curve on which solutions to a DE have the same constant slope. As an example, for the DE $y' = t + y$, if we set $y' = t + y = 1$ we see that the line $y = -t + 1$ is an isocline on which solutions have slope 1. Isoclines are useful for sketching slope fields quickly.

An **equilibrium solution** to a DE is a constant solution, found by setting $y' = f(t, y) = 0$.

A DE that can be written in the form $N(y)y' = M(x)$ is called **separable**. Sometimes a DE is not separable, but a substitution will turn it in to a separable DE.

An **explicit solution** to a DE is a a solution given in the form $y = y(t)$. An **implicit solution** is one given by an equation not in this form.

A **first order linear DE** can be written in the form $y' + p(t)y = q(t)$. To solve such a DE, use the integrating factor method:

1. Make sure the DE is in the form above (nothing in front of y').

2. Find the **integrating factor** $\mu(t) = \int p(t)\,\mathrm{d}t$.

3. Multiply the DE by $\mu(t)$ and check that the left hand side becomes $(\mu(t)y)'$.

4. Integrate both sides, getting $\mu(t)y$ on the left and the constant of integration on the right.

5. Solve for $y(t)$.

An **initial value problem (IVP)** is a DE $y' = f(t, y)$ with an **initial condition** $y(t_0) = y_0$.

Existence/Uniqueness (E/U) Theorem: For a first order linear DE $y' + p(t)y = q(t)$ with initial condition $y(t_0) = y_0$, there exists a unique solution to the IVP on the interval $I = (a, b)$ containing t_0 if $p(t)$ and $q(t)$ are continuous on I.

The largest interval on which the solution is valid and contains t_0 is called the **interval of validity**.

Notice that DEs like $y' = \sqrt{y}$ can't be written in this form, so the E/U theorem doesn't apply.

A first order DE is **autonomous** if it can be written in the form $y' = f(y)$ (no dependence on t).

An **(asymptotically) stable** equilibrium solution is one that nearby solutions approach as $t \to \infty$. If this nearby solutions move away from an equilibrium solution as $t \to \infty$, an equilibrium solution is called **unstable**. If nearby solutions on one side approach an equilibrium solution and move away on the other side, the equilibrium solution is called **semi-stable**.

An **exponential growth/decay equation** is of the form $y' = ry$. This is a separable DE with solution $y = y_0e^{rt}$.

The **logistic equation** is $y' = ry(1 - \frac{y}{K})$ where $r, K > 0$. Here r is the growth rate and K is the limiting value. One can show that the solution to this DE is $y = \frac{y_0K}{y_0+(K-y_0)e^{-rt}}$.

A **threshold equation** is of the form $y' = -ry(1 - \frac{y}{T})$ where $r, T > 0$. As t increases, y either approaches 0 or grows without bound, depending on whether the initial value is less than or greater than the threshold value T.

Mixing problems: know how to set up a DE of the form $q' = \text{flow rate in} - \text{flow rate out}$.

The DE $m\frac{dv}{dt} = mg - kv$ models a falling object, where m is the mass of the object, g is the acceleration due to gravity ($g = 9.8$ m/s^2 $= 32$ ft/s^2) and $k > 0$ is the constant of proportionality slowing the object down (drag). Note that this DE is set up so that downward is the positive direction. We can model similar situations with a similar DE by taking into consideration the signs of mg and kv, and the choice of positive direction being up or down.