In section we showed that $A = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}$ is diagonalizable, and $A = PDP^{-1}$, where $D = \begin{bmatrix} 3 & 0 \\ 0 & -7 \end{bmatrix}$, $P = \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix}$, and $P^{-1} = \begin{bmatrix} 3/10 & 1/10 \\ -1/10 & 3/10 \end{bmatrix}$.

The columns of $P$ form a basis for $\mathbb{R}^2$ we'll call $B$, so $B = \left\{ \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \end{bmatrix} \right\}$.

**The mappings $\mathbf{x} \mapsto A\mathbf{x}$ and $\mathbf{u} \mapsto D\mathbf{u}$ describe the same linear transformation, relative to the bases $S$ and $B$ respectively.**

Let's see what this means by considering the vector $\mathbf{x} = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$.

First note that $A\mathbf{x} = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix} \begin{bmatrix} 4 \\ 8 \end{bmatrix} = \begin{bmatrix} 32 \\ -36 \end{bmatrix}$

Next, we know that the coordinate vector of $\mathbf{x}$ relative to the basis $B$ is

$[\mathbf{x}]_B = P^{-1}\mathbf{x} = \begin{bmatrix} 3/10 & 1/10 \\ -1/10 & 3/10 \end{bmatrix} \begin{bmatrix} 4 \\ 8 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$.

If we compute $D[\mathbf{x}]_B$ we have $D[\mathbf{x}]_B = \begin{bmatrix} 3 & 0 \\ 0 & -7 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ -14 \end{bmatrix}$.

Notice that $\begin{bmatrix} 6 \\ -14 \end{bmatrix}$ is the coordinate vector of $A\mathbf{x} = \begin{bmatrix} 32 \\ -36 \end{bmatrix}$ relative to the basis $B$, since $A\mathbf{x} = P[A\mathbf{x}]_B = \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 6 \\ -14 \end{bmatrix} = \begin{bmatrix} 32 \\ -36 \end{bmatrix}$.

**Thus the output of $\mathbf{x} \mapsto A\mathbf{x}$ and $\mathbf{u} \mapsto D\mathbf{u}$ are the same, relative to different bases.**

Note that $D = P^{-1}AP$, and observe that $P$ is the change-of-coordinate matrix that converts vectors from $B$ to $S$, and $P^{-1}$ converts vectors from $S$ to $B$.

This means we can see how $\mathbf{u} \mapsto D\mathbf{u} = P^{-1}AP\mathbf{u}$ works in three steps: $P$ converts the vector from $B$ to $S$, then we multiply by $A$, and lastly $P^{-1}$ converts the resulting vector from $S$ back to $B$, as follows:

First, $P$ converts $[\mathbf{x}]_B = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ from $B$ to $S$: $P[\mathbf{x}]_B = \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \end{bmatrix} = \mathbf{x}$ as above.

Next multiply the result by $A$: $A\mathbf{x} = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix} \begin{bmatrix} 4 \\ 8 \end{bmatrix} = \begin{bmatrix} 32 \\ -36 \end{bmatrix}$ as above.

Lastly, $P^{-1}$ converts $A\mathbf{x}$ from $S$ to $B$: $P^{-1}A\mathbf{x} = \begin{bmatrix} 3/10 & 1/10 \\ -1/10 & 3/10 \end{bmatrix} \begin{bmatrix} 32 \\ -36 \end{bmatrix} = \begin{bmatrix} 6 \\ -14 \end{bmatrix}$.