Suppose the characteristic equation for the DE

$$ay'' + by' + cy = 0$$

has a pair of complex conjugate roots $r_1, r_2 = \alpha \pm \beta i$. Then

$$y_1(t) = e^{r_1 t} = e^{(\alpha+\beta i)t} = e^{\alpha t} e^{i\beta t}$$

and

$$y_2(t) = e^{r_2 t} = e^{(\alpha-\beta i)t} = e^{\alpha t} e^{-i\beta t}$$

are two solutions. To get two real solutions, we use Euler's formula

$$e^{it} = \cos(t) + i\ \sin(t)$$

and

$$e^{-it} = \cos(t) - i\ \sin(t)$$

. So we rewrite

$$y_1(t) = e^{\alpha t} e^{i\beta t} = e^{\alpha t}(\cos(\beta t) + i\ \sin(\beta t))$$

and

$$y_2(t) = e^{\alpha t} e^{i\beta t} = e^{\alpha t}(\cos(\beta t) - i\ \sin(\beta t)).$$

Then notice that

$$f(t) = \frac{1}{2}(y_1(t) + y_2(t)) = e^{\alpha t}\cos(\beta t)$$

and

$$g(t) = \frac{1}{2i}(y_1(t) - y_2(t)) = e^{\alpha t}\sin(\beta t).$$

This leads to the general solution

$$c_1 f(t) + c_2 g(t) = c_1 e^{\alpha t}\cos(\beta t) + c_2 e^{\alpha t}\sin(\beta t) = e^{\alpha t}(c_1\cos(\beta t) + c_2\sin(\beta t).$$

But where does Euler's formula come from?

From calculus, any infinitely differentiable function $f(t)$ can be written as $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} t^n$, called the Taylor series at 0 or Maclaurin series. One can show from this that

$$e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!}, \qquad \cos(t) = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n)!}, \qquad \sin(t) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} t^{2n-1}}{(2n-1)!}.$$

Substituting it for t gives

$$e^{it} = \sum_{n=0}^{\infty} \frac{(it)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n)!} + i \sum_{n=1}^{\infty} \frac{(-1)^{n-1} t^{2n-1}}{(2n-1)!} = \cos(t) + i\ \sin(t)$$

by separating real and imaginary parts and using the fact that $i^2 = -1, i^3 = -1, i^4 = 1$ and so on.