If A is not diagonalizable, maybe we can write $A = PJP^{-1}$ for an "almost diagonal" matrix J. This leads to the following theorem.

Theorem. Suppose that A is a 2×2 matrix with repeated eigenvalue λ whose eigenspace is only one-dimensional, say $E_\lambda = \text{Span}\{\mathbf{v_1}\}$. Let $\mathbf{v_2}$ be a solution to $(A - \lambda I)\mathbf{x} = \mathbf{v_1}$. Let $J = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$ and $P = \begin{bmatrix} \mathbf{v_1} & \mathbf{v_2} \end{bmatrix}$. Then $A = PJP^{-1}$.

Proof:

The proof that P is invertible is left for the reader. Note that
$A = PJP^{-1} \Leftrightarrow AP = PJ \Leftrightarrow A\begin{bmatrix} \mathbf{v_1} & \mathbf{v_2} \end{bmatrix} = \begin{bmatrix} \mathbf{v_1} & \mathbf{v_2} \end{bmatrix} \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$.

Thus the columns of P must satisfy $A\mathbf{v_1} = \lambda\mathbf{v_1}$ and $A\mathbf{v_2} = \lambda\mathbf{v_2} + \mathbf{v_1}$.

Rearranging, this becomes $A\mathbf{v_1} - \lambda\mathbf{v_1} = \mathbf{0}$ and $A\mathbf{v_2} - \lambda\mathbf{v_2} = \mathbf{v_1}$,

or $(A - \lambda I)\mathbf{v_1} = \mathbf{0}$ and $(A - \lambda I)\mathbf{v_2} = \mathbf{v_1}$.

The first equation says that $\mathbf{v_1}$ is an eigenvalue for A and the second equation says that $\mathbf{v_2}$ is a solution to $(A - \lambda I)\mathbf{x} = \mathbf{v_1}$, as assumed.