① Using Riemann Sums, find the limit of

$$(2^2 3^3 \dots n^n)^{\frac{1}{n^2 \ln(n)}}.$$

<u>Solution:</u>

$$\lim_{n\to\infty} (2^2 3^3 \dots n^n)^{\frac{1}{n^2 \ln(n)}}$$

$$= e^{\lim_{n\to\infty} \ln\left[(2^2 3^3 \dots n^n)\right]^{\frac{1}{n^2 \ln(n)}}}$$

$$= e^{\lim_{n\to\infty} \frac{1}{n^2 \ln(n)} \ln(2^2 3^3 \dots n^n)}$$

$$= e^{\lim_{n\to\infty} \frac{1}{n^2 \ln(n)}\left[\ln(2^2) + \ln(3^3) + \dots + \ln(n^n)\right]}$$

$$= e^{\lim_{n\to\infty} \frac{1}{n^2 \ln(n)}\left[2\ln 2 + 3\ln 3 + \dots + n\ln(n)\right]}$$

$$= e^{\lim_{n\to\infty} \frac{1}{n^2 \ln n} \cdot \sum_{K=1}^{n} K \ln K}$$ (starts at $K=1$ since $1\ln(1) = 0$)

$$= e^{\lim_{n\to\infty} \frac{1}{\ln n} \cdot \frac{1}{n} \cdot \sum_{K=1}^{n} \frac{K}{n} \ln K}$$

$$= e^{\lim_{n\to\infty} \frac{1}{\ln n} \cdot \frac{1}{n} \cdot \sum_{K=1}^{n} \frac{K}{n}\left(\ln \frac{K}{n} + \ln n\right)},$$ (since $\ln \frac{K}{n} = \ln K - \ln n$ $\longrightarrow \ln K = \ln \frac{K}{n} + \ln n$)

$$= e^{\lim_{n\to\infty} \frac{1}{\ln n} \cdot \frac{1}{n} \cdot \sum_{K=1}^{n} \frac{K}{n}\ln \frac{K}{n} + \frac{K}{n}\ln n}$$

$$= e^{\lim_{n\to\infty} \frac{1}{\ln n} \cdot \frac{1}{n}\left(\sum_{k=1}^{n} \frac{k}{n}\ln\frac{k}{n} + \sum_{k=1}^{n}\frac{k}{n}\ln n\right)}$$ $\left(\text{since } \sum a_k + b_k = \sum a_k + \sum b_k\right)$

Now we leave the $\frac{1}{\ln n}$ outside the 1st sum ~~and~~ but distribute it to the 2nd sum to get: (and distribute the $\frac{1}{n}$)

$$= e^{\lim_{n\to\infty} \frac{1}{\ln n}\left(\frac{1}{n}\sum_{k=1}^{n}\frac{k}{n}\ln\frac{k}{n}\right) + \frac{1}{n}\sum_{k=1}^{n}\frac{k}{n}\frac{\cancel{\ln n}}{\cancel{\ln n}}}$$

$$= e^{\lim_{n\to\infty} \frac{1}{\ln n}\underbrace{\left(\int_0^1 x\ln x\,dx\right)}_{\text{I}} + \underbrace{\int_0^1 x\,dx}_{\text{II}}}$$

I: $\int_0^1 x\ln x\,dx = \left[\frac{x^2}{2}\ln x\right]_0^1 - \frac{1}{2}\int_0^1 x\,dx$

$\boxed{\begin{array}{ll} u = \ln x & v = \frac{t^2}{2} \\ du = \frac{1}{x}dx & dv = x\,dx \end{array}}$ $= \lim_{t\to 0^+}\left[\frac{x^2}{2}\ln x\right]_t^1 - \left[\frac{1}{4}x^2\right]_0^1$

$= \lim_{t\to 0^+} -\frac{t^2}{2}\ln t - \frac{1}{4}$

$= 0 - \frac{1}{4} = \frac{-1}{4}$ (by l'Hopital, see below)

II $\int_0^1 x\,dx = \left[\frac{x^2}{2}\right]_0^1 = \frac{1}{2}$

So the answer is $e^{\lim_{n\to\infty}\frac{1}{\ln n}\cdot\left(\frac{-1}{4}\right) + \frac{1}{2}}$

$= e^{\lim_{n\to\infty}\frac{1}{2}}$ $\left(\text{since } \frac{1}{\ln n}\to 0 \text{ as } n\to\infty\right)$

$= \boxed{e^{\frac{1}{2}}}$

L'hopital's rule:

$$\lim_{t\to 0^+} t^2 \ln t = \lim_{t\to 0^+} \frac{\ln t}{\frac{1}{t^2}} \quad \left(\frac{-\infty}{\infty}\right)$$

l'Hopital

$$\downarrow$$

$$= \lim_{t\to 0^+} \frac{\frac{1}{t}}{\frac{-2}{t^3}} \quad \left(\text{take derivatives of top + bottom}\right)$$

$$= \lim_{t\to 0^+} \frac{1}{t} \cdot \frac{t^3}{-2}$$

$$= \frac{-1}{2} \lim_{t\to 0^+} t^2 = 0.$$

(2) Using Riemann sums, find the limit of

$$e^{\frac{n}{4}} n^{\left(-\frac{n+1}{2}\right)} \left(2^2 3^3 \cdots n^n\right)^{\frac{1}{n}}.$$

Solution:

$$\lim_{n\to\infty} e^{\frac{n}{4}} n^{\left(-\frac{n+1}{2}\right)} \left(2^2 3^3 \cdots n^n\right)^{\frac{1}{n}}$$

$$= e^{\lim_{n\to\infty} \ln\left[e^{\frac{n}{4}} \cdot n^{-\frac{n+1}{2}} \left(2^2 3^3 \cdots n^n\right)^{\frac{1}{n}}\right]}$$

$$= e^{\lim_{n\to\infty} \ln e^{\frac{n}{4}} + \ln n^{-\frac{n+1}{2}} + \frac{1}{n}\left(\ln 2^2 + \ln 3^3 + \dots + \ln n^n\right)}$$

$$= e^{\lim_{n\to\infty} \frac{n}{4} - \frac{n+1}{2} \ln n + \frac{1}{n} \sum_{K=1}^{n} K \ln K}$$

$$= e^{\lim_{n\to\infty} \frac{n}{4} - \frac{n+1}{2}\ln n + \frac{1}{n}\sum_{K=1}^{n} K \ln\frac{K}{n} + \frac{1}{n}\sum_{K=1}^{n} K \ln n}$$

(using $\ln K = \ln\frac{K}{n} + \ln n$)

$$= e^{\lim_{n\to\infty} \frac{n}{4} - \frac{n+1}{2}\ln n + n\cdot\frac{1}{n}\sum_{K=1}^{n} \frac{K}{n}\ln\frac{K}{n} + n\ln n\cdot\frac{1}{n}\sum_{K=1}^{n}\frac{K}{n}\frac{\cancel{\ln n}}{\cancel{\ln n}}}$$

$$= e^{\lim_{n\to\infty} \frac{n}{4} - \frac{n+1}{2}\ln n + n\left(\int_0^1 x\ln x\,dx\right) + n\ln n\left(\int_0^1 x\,dx\right)}$$

$$= e^{\lim_{n\to\infty} \frac{n}{4} - \frac{n+1}{2}\ln n + n\left(\frac{-1}{4}\right) + n\ln n\left(\frac{1}{2}\right)}$$ (by previous problem)

$$= e^{\lim_{n\to\infty} \cancel{\frac{n}{4}} - \frac{n+1}{2}\ln n - \cancel{\frac{n}{4}} + \frac{n}{2}\ln n}$$

$$= e^{\lim_{n\to\infty} \frac{-1}{2}\ln n}$$

$$= e^{\lim_{n\to\infty} \ln n^{\frac{-1}{2}}}$$

$$= \lim_{n\to\infty} \cancel{e}^{\cancel{\ln}\, n^{\frac{-1}{2}}}$$

$$= \lim_{n\to\infty} n^{\frac{-1}{2}} = \lim_{n\to\infty} \frac{1}{n^{\frac{1}{2}}} = 0.$$