Problem: Find the general solution to

$$y'' - y = t^2 e^{-t}.$$

Step 1. We found that

$$y_h = c_1 e^t + c_2 e^{-t}.$$

Step 2. We first guessed

$$y_p = (At^2 + Bt + C)e^{-t},$$

which needs to be modified, for the following reason:
Notice that if we multiplied the e^{-t} through the parentheses above, one of the terms of our y_p would be the same as the second term of our y_h.
This is why we multiply our whole y_p by t, so our new guess is

$$y_p = (At^3 + Bt^2 + Ct)e^{-t}.$$

Note we no longer have this problem with our new y_p.
Next, we calculate $y_p'' = ((At^3 + (-6A + B)t^2 + (6A - 4B + C)t + 2B - 2C))e^{-t}$.
After plugging into our DE and simplifying, we have

$$y_p'' - y_p = (-6At^2 + (6A - 4B)t + 2B - 2C)e^{-t} = t^2 e^{-t}.$$

We now can equate coefficients according to powers of t (distribute the exponential on the left to see this) to get a system of 3 equations and 3 unknowns,
$-6A = 1$
$6A - 4B = 0$
$2B - 2C = 0,$
which we can solve to find $A = -1/6, B = -1/4, C = -1/4$.

Note: Had we stuck with our original y_p and worked this out, we'd end up with something ridiculous like $e^{-t} = 0$. Any time this happens, it means you should go back and compare your y_h and y_p, and multiply your y_p by at least one factor of t (or as in the next example, at least part of your y_p).

Step 3.
Thus our solution is

$$y = y_h + y_p = c_1 e^t + c_2 e^{-t} + (\frac{-1}{6}t^3 - \frac{1}{4}t^2 - \frac{1}{4}t)e^{-t}.$$

Here is another example to clarify the process going on above:
Find the general solution to

$$y'' - 100y = 9t^2e^{10t} + cos(t).$$

Step 1. We find that

$$y_h = c_1e^{10t} + c_2e^{-10t}.$$

Step 2. We first guess

$$y_p = (At^2 + Bt + C)e^{10t} + Dcos(t) + Esin(t).$$

Notice that if we multiplied the exponential term through the parentheses, we would end up getting part of the homogeneous solution showing up. Since the problem arises from the first term the whole first term will get multiplied by t. The second and third terms are okay as they are.

The correct guess for the form of the particular solution is then

$$y_p = t(At^2 + Bt + C)e^{10t} + Dcos(t) + Esin(t).$$

You can finish this one for fun.