CONGRUENCE OF HERMITIAN MATRICES BY HERMITIAN MATRICES [*]

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Abstract. Two nonsingular Hermitian matrices $A, B \in M_n(\mathbb{C})$ are said to be Hermitian-congruent if there exists a Hermitian matrix $C \in M_n(\mathbb{C})$ such that $B = CAC$. In this paper, we give necessary and sufficient conditions for two simultaneously unitarily diagonalizable Hermitian matrices A and B to be Hermitian-congruent. Moreover, when A and B are Hermitian-congruent, we describe the possible inertias of the Hermitian matrices C that carry the congruence. We also give necessary and sufficient conditions for any 2-by-2 Hermitian matrices to be Hermitian-congruent. In both of the studied cases, we show that if A and B are real and Hermitian-congruent, then they are congruent by a real symmetric matrix. Finally we prove that if A and B are 2-by-2 real symmetric matrices having the same sign pattern, then there is always a real symmetric matrix C satisfying $B = CAC$. Moreover, if both matrices are positive, then C can be picked with arbitrary inertia.

Key words. Congruence, Hermitian matrix, simultaneously unitarily diagonalizable, sign pattern.

AMS subject classifications. 15A21, 15A24, 15A48, 15A57

1. Introduction.

Matrices $A, B \in M_n(\mathbb{C})$ (M_n for short) are said to be congruent if there is a nonsingular matrix $C \in M_n(\mathbb{C})$ such that $B = C^*AC$. Congruence is an equivalence relation on $M_n(\mathbb{C})$. If * is replaced by ^{T} and the matrices are real, then congruence is also an equivalence relation on $M_n(\mathbb{R})$. If A and B are Hermitian, it is well known that they are congruent if and only if they have the same inertia (number of positive, negative and zero eigenvalues, counting multiplicities) [3]. In particular, if A and B are real symmetric and have the same inertia, they are congruent by a real matrix [3]. We are interested here in the case in which A and B are Hermitian (real symmetric) and C can be chosen Hermitian (real symmetric), as well.

It is a notable fact that if A and B are positive definite, then there is a unique positive definite matrix C such that $B = CAC$ [2]; the formula for C is

$$C = A^{-1/2}(A^{1/2}BA^{1/2})^{1/2}A^{-1/2}.$$

Moreover, as follows from Theorem 2.3, A and B are congruent by a Hermitian matrix with arbitrary inertia. Since $B = CAC$ if and only if $-B = C(-A)C$, similar observations may be made when A and B are negative definite. However, if A and B are congruent but not positive (negative) definite, it may happen that, not only there is no positive definite C that carries the congruence, but no Hermitian C at all. For example, let

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}.$$

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We consider here the question of which pairs of nonsingular Hermitian (real symmetric) matrices A and B are congruent by a Hermitian (real symmetric) C and what are the possible inertias of C. The order in which A and B are taken is immaterial.

We develop some general theory for this question and then specialize it to give some explicit results in certain cases. In particular, we solve our question when A and B are simultaneously unitarily similar to diagonal matrices. Also, when $n = 2$ highly explicit results are given, including remarkable results involving the sign patterns of A and B. In view of the facts mentioned for A and B positive (negative) definite, if convenient, we may concentrate upon the cases in which A and B are indefinite.

Notice that the equation $B = XAX$ is a particular case of the general continuous algebraic Ricatti equation [4]

$$XAX + XD + D^*X - B = 0, \tag{1.1}$$

when $D = 0$. Ricatti equations are of great interest because of their important role in optimal filter design and control theory. In many applications, Hermitian solutions of (1.1) are required. Most of the results in the literature on this problem assume that A and B are positive semidefinite matrices. Some results can also be found for indefinite matrices assuming that A satisfies some conditions motivated by control theory [1, 5].

2. Notation and general results.

We start this section with some notation and definitions that will be needed throughout the paper.

Let $A, B \in M_n$ be Hermitian matrices. We say that A and B are *Hermitian-congruent* if there is a nonsingular Hermitian matrix $X \in M_n$ such that $B = XAX$.

We say that $K \in M_n$ is a *signature* matrix if K is a diagonal matrix with eigenvalues in $\{-1, 1\}$. If K is a signature matrix, then $W \in M_n$ is said to be *K-unitary* if $W^*KW = K$. It is easy to see that the set of all K-unitary matrices form a group under multiplication.

If $A = [a_{ij}] \in M_n$, we say that $S = [s_{ij}] \in M_n$ is the *sign matrix* of A, and we write $S = sign(A)$, if $s_{ij} = 1$ for $a_{ij} > 0$; $s_{ij} = -1$ for $a_{ij} < 0$, and $s_{ij} = 0$ for $a_{ij} = 0$, $i, j = 1, ..., n$.

The next two results allow us to get equivalent statements of our problem.

THEOREM 2.1. *Let $A, B \in M_n$ be nonsingular Hermitian matrices with the same inertia. Let $Y, Z \in M_n$ be such that $A = Y^*KY$ and $B = Z^*KZ$, respectively, where K is a signature matrix with the same inertia as A. Let $X \in M_n$ be a Hermitian matrix. Then, $B = XAX$ if and only if $X = Y^{-1}WZ$ for some K-unitary matrix $W \in M_n$.*

Proof. Suppose that there is a Hermitian matrix $X \in M_n$ such that $B = XAX$. Then,

$$Z^*KZ = X(Y^*KY)X,$$

or, equivalently,

$$K = (Z^{-*}XY^*)K(YXZ^{-1}).$$

Thus, $W = YXZ^{-1}$ is a K-unitary matrix.

Now suppose that $X = Y^{-1}WZ$ for some K-unitary matrix $W \in M_n$. Since X is Hermitian,

$$Y^{-1}WZ = Z^*W^*Y^{-*}$$

Then,

$$XAX = (Z^*W^*Y^{-*})A(Y^{-1}WZ) = B.$$

□

COROLLARY 2.2. *Let $A, B \in M_n$ be nonsingular Hermitian matrices. Let $Y \in M_n$ be such that $A = Y^*KY$, where K is a signature matrix with the same inertia as A. There is a Hermitian matrix $X \in M_n$ such that $B = XAX$ if and only if there is a matrix $Z \in M_n$ such that the following conditions are satisfied:*

1. *$B = Z^*KZ$,*
2. *ZY^* is a Hermitian matrix.*

Proof. Suppose that there is a Hermitian matrix $X \in M_n$ such that $B = XAX$. Since A and B have the same inertia, there is $\tilde{Z} \in M_n$ such that $B = \tilde{Z}^*K\tilde{Z}$. By Theorem 2.1,

$$X = Y^{-1}W\tilde{Z},$$

for some K-unitary matrix $W \in M_n$. Since X is Hermitian,

$$Y^{-1}W\tilde{Z} = \tilde{Z}^*W^*Y^{-*}. \tag{2.1}$$

Let $Z := W\tilde{Z}$. Then, using (2.1),

$$ZY^* = W\tilde{Z}Y^* = Y\tilde{Z}^*W^* = YZ^*$$

Clearly, conditions 1. and 2. hold with $Z := W\tilde{Z}$.

Now suppose that there exists a nonsingular matrix $Z \in M_n$ such that $B = Z^*KZ$ and ZY^* is Hermitian. Then, $X = Y^{-1}Z$ is a Hermitian solution of $B = XAX$. □

We now show that if two Hermitian matrices are congruent by a definite matrix then they are congruent by a Hermitian matrix with any inertia.

THEOREM 2.3. *Let $A, B \in M_n$ be Hermitian matrices. If the matrix equation $B = XAX$ has a definite solution, then it has a Hermitian solution with an arbitrary inertia.*

Proof. We can assume that X is a positive definite solution of $B = XAX$. If not, then $-X$ is a positive definite solution. Then, there exists a nonsingular matrix $S \in M_n$ such that $X = S^*S$. Thus, we have

$$S^*SAS^*S = B,$$

or equivalently,

$$SAS^* = S^{-*}BS^{-1}.$$

Since SAS^* is Hermitian, it is unitarily diagonalizable. Let $U \in M_n$ be a unitary matrix and D a diagonal matrix such that $U^*DU = SAS^*$. Then,

$$USAS^*U^* = D = US^{-*}BS^{-1}U^*.$$

Let K be any $n \times n$ signature matrix. Since D is diagonal, $KDK = D$, and therefore,

$$KUSAS^*U^*K = D = US^{-*}BS^{-1}U^*,$$

or equivalently,

$$(S^*U^*KUS)A(S^*U^*KUS) = B,$$

and the result follows. □

We finish this section with a lemma that facilitates the proofs of our results, as it allows us to assume that A and/or B have a particular form.

LEMMA 2.4. *Let $A, B \in M_n$ be Hermitian matrices and let $C \in M_n$ be a nonsingular matrix. Then A and B are Hermitian-congruent if and only if C^*AC and $C^{-1}BC^{-*}$ are Hermitian-congruent.*

Proof. It suffices to prove that if A and B are Hermitian-congruent then C^*AC and $C^{-1}BC^{-*}$ are Hermitian-congruent. Let $X \in M_n$ be a nonsingular Hermitian matrix such that $B = XAX$. Then

$$C^{-1}BC^{-*} = \left(C^{-1}XC^{-*}\right)\left(C^*AC\right)\left(C^{-1}XC^{-*}\right). \tag{2.2}$$

Clearly, $C^{-1}XC^{-*}$ is Hermitian. □

3. The simultaneously unitarily diagonalizable case. In this section we study the existence of a Hermitian matrix $X \in M_n$ such that $B = XAX$, when $A, B \in M_n$ are nonsingular simultaneously unitarily diagonalizable matrices.

We first consider the case in which A and B are real diagonal matrices. We may assume that A is a signature matrix, which simplifies our calculations. The transition to the case in which A is a general diagonal matrix follows easily from the next result, whose proof is similar to the proof of Lemma 2.4, taking into account that $A = |A|^{\frac{1}{2}}K|A|^{\frac{1}{2}}$, with $K = sign(A)$.

PROPOSITION 3.1. *Let $A, B \in M_n$ be real nonsingular diagonal matrices and let K be the sign matrix of A. Let $X \in M_n$. Then $B = XAX$ if and only if $B|A| = (|A|^{\frac{1}{2}}X|A|^{\frac{1}{2}})K(|A|^{\frac{1}{2}}X|A|^{\frac{1}{2}})$.*

We now give some lemmas for real diagonal matrices A and B that will be needed in the proof of our main result. We start with a result that shows that we can reduce our problem to the cases in which $sign(A) = sign(B)$ and $sign(A) = -sign(B)$.

LEMMA 3.2. *Let*

$$A = A_1 \oplus (-A_2) \oplus A_3 \oplus (-A_4) \tag{3.1}$$

and

$$B = (-B_1) \oplus B_2 \oplus B_3 \oplus (-B_4), \tag{3.2}$$

where $A_1, B_1, A_2, B_2 \in M_q$, $A_3, B_3 \in M_p$ and $A_4, B_4 \in M_r$ are positive definite diagonal matrices. Let $X \in M_{2q+p+r}$ be a Hermitian matrix. Then X is a solution of $B = XAX$ if and only if $X = X_1 \oplus X_2$, where $X_1 \in M_{2q}$ and $X_2 \in M_{p+r}$ are such that

$$X_1(A_1 \oplus (-A_2))X_1 = (-B_1) \oplus B_2,$$

$$X_2(A_3 \oplus (-A_4))X_2 = B_3 \oplus (-B_4).$$

Proof. Bearing in mind Proposition 3.1, we assume, without loss of generality, that

$$A = I_q \oplus (-I_q) \oplus I_p \oplus (-I_r)$$

and

$$B = (-D_1) \oplus D_2 \oplus D_3 \oplus (-D_4), \tag{3.3}$$

where $D_i = A_i B_i$, $i = 1, 2, 3, 4$. Suppose that

$$X = \begin{bmatrix} X_{11} & X_{12} & X_{13} & X_{14} \\ X_{12}^* & X_{22} & X_{23} & X_{24} \\ X_{13}^* & X_{23}^* & X_{33} & X_{34} \\ X_{14}^* & X_{24}^* & X_{34}^* & X_{44} \end{bmatrix}, \tag{3.4}$$

where $X_{11}, X_{22} \in M_q$, $X_{33} \in M_p$ and $X_{44} \in M_r$, is a Hermitian matrix such that $B = XAX$. Then,

$$X^{-1} = B^{-1}XA = \begin{bmatrix} -D_1^{-1}X_{11} & D_1^{-1}X_{12} & -D_1^{-1}X_{13} & D_1^{-1}X_{14} \\ D_2^{-1}X_{12}^* & -D_2^{-1}X_{22} & D_2^{-1}X_{23} & -D_2^{-1}X_{24} \\ D_3^{-1}X_{13}^* & -D_3^{-1}X_{23}^* & D_3^{-1}X_{33} & -D_3^{-1}X_{34} \\ -D_4^{-1}X_{14}^* & D_4^{-1}X_{24}^* & -D_4^{-1}X_{34}^* & D_4^{-1}X_{44} \end{bmatrix}. \tag{3.5}$$

Since X^{-1} is Hermitian, it follows that

$$D_i^{-1}X_{ij} = -X_{ij}D_j^{-1}, \qquad \text{for } i \in \{1, 2\},\ j \in \{3, 4\}. \tag{3.6}$$

As the main diagonal of D_i is positive, for $i = 1, 2, 3, 4$, condition (3.6) implies that $X_{ij} = 0$ for $i \in \{1, 2\}$, $j \in \{3, 4\}$, and the result follows.

The proof of the converse is trivial. □

The next lemma considers the case in which $A, B \in M_n$ are real nonsingular diagonal matrices with the same sign matrix.

LEMMA 3.3. *Let $A, B \in M_n$ be real nonsingular diagonal matrices such that $sign(A) = sign(B)$. Then there is a real diagonal matrix $X \in M_n$, with arbitrary inertia, such that $B = XAX$.*

Proof. If $sign(A) = sign(B)$, there exists a signature matrix K such that $A = |A|^{1/2}K|A|^{1/2}$ and $B = |B|^{1/2}K|B|^{1/2}$. For an arbitrary signature matrix $T \in M_n$, since $T^2 = I_n$, we have

$$\begin{aligned} B &= |B|^{1/2}K|B|^{1/2} \\ &= \left(|B|^{1/2}|A|^{-1/2}T\right)\left(|A|^{1/2}K|A|^{1/2}\right)\left(|B|^{1/2}|A|^{-1/2}T\right) \\ &= \left(|B|^{1/2}|A|^{-1/2}T\right)A\left(|B|^{1/2}|A|^{-1/2}T\right). \end{aligned}$$

Clearly, $|B|^{1/2}|A|^{-1/2}T$ is real diagonal with the same inertia as T. □

The next two lemmas consider the case in which $A, B \in M_n$ are real nonsingular diagonal matrices such that $sign(A) = -sign(B)$.

LEMMA 3.4. *Let $\lambda > 0$. Then there is a real symmetric matrix $X \in M_{2s}$ such that*

$$X(I_s \oplus (-I_s))X = \lambda((-I_s) \oplus I_s). \tag{3.7}$$

Moreover, if X is any Hermitian solution to (3.7), then X has exactly s positive eigenvalues.

Proof. The matrix

$$X = \sqrt{\lambda}\begin{bmatrix} 0 & I_s \\ I_s & 0 \end{bmatrix} \tag{3.8}$$

proves the first part of the statement. To prove the second part of the statement, suppose that (3.7) holds for some Hermitian matrix $X \in M_{2s}$. Let $Y = 1/\sqrt{\lambda}X$. Then,

$$Y\begin{bmatrix} I_s & 0 \\ 0 & -I_s \end{bmatrix} = \begin{bmatrix} -I_s & 0 \\ 0 & I_s \end{bmatrix} Y^{-1}, \tag{3.9}$$

which is equivalent to

$$Y^{-1} = \begin{bmatrix} -I_s & 0 \\ 0 & I_s \end{bmatrix} (-Y) \begin{bmatrix} -I_s & 0 \\ 0 & I_s \end{bmatrix} \tag{3.10}$$

Thus, Y^{-1} is similar to $-Y$, which implies that the number of positive and negative eigenvalues of Y, and, therefore, of X, are the same and equal to s. □

LEMMA 3.5. *Let*

$$A = A_1 \oplus (-A_2) \qquad \text{and} \qquad B = (-B_1) \oplus B_2, \tag{3.11}$$

where $A_1, A_2, B_1, B_2 \in M_q$ are positive definite diagonal matrices. If there is a Hermitian matrix $X \in M_n$ such that $B = XAX$, then A_1B_1 and A_2B_2 are similar and X has q positive eigenvalues. Conversely, if A_1B_1 and A_2B_2 are similar, then there is a real symmetric matrix $X \in M_n$ with q positive eigenvalues such that $B = XAX$.

Proof. Taking into account Proposition 3.1, we assume, without loss of generality, that

$$A = I_q \oplus (-I_q) \qquad \text{and} \qquad B = (-D_1) \oplus D_2,$$

where $D_1 = A_1B_1$ and $D_2 = A_2B_2$. Suppose that

$$X = \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^* & X_{22} \end{bmatrix}, \tag{3.12}$$

with $X_{11}, X_{22} \in M_q$, is a Hermitian matrix such that $B = XAX$. Then,

$$X^{-1} = B^{-1}XA = \begin{bmatrix} -D_1^{-1}X_{11} & D_1^{-1}X_{12} \\ D_2^{-1}X_{12}^* & -D_2^{-1}X_{22} \end{bmatrix}. \tag{3.13}$$

Because X^{-1} is Hermitian,

$$\begin{aligned} D_1^{-1}X_{11} &= X_{11}D_1^{-1}, \\ D_2^{-1}X_{22} &= X_{22}D_2^{-1} \\ D_1^{-1}X_{12} &= X_{12}D_2^{-1}, \end{aligned}$$

which is equivalent to

$$(X_{ij})_{kl} = 0 \text{ or } \lambda_{ki} = \lambda_{lj}, \qquad k, l = 1, \ldots, q, \text{ and } i, j = 1, 2, \; i \leq j, \tag{3.14}$$

where λ_{r1} and λ_{r2} denote the rth entry on the main diagonal of D_1 and D_2, respectively.
Let $P \in M_{2q}$ be a permutation matrix such that

$$P^T \begin{bmatrix} -D_1 & 0 \\ 0 & D_2 \end{bmatrix} P = \begin{bmatrix} R & 0 & 0 \\ 0 & -R_{44} & 0 \\ 0 & 0 & R_{55} \end{bmatrix}, \tag{3.15}$$

where $R_{44}, R_{55} \in M_w$ are positive definite diagonal matrices such that the eigenvalues of R_{44} (resp. R_{55}) are precisely the eigenvalues of D_1 (resp. D_2) that are not eigenvalues of D_2 (resp. D_1), considering multiple eigenvalues (note that R_{44} and R_{55} have no common eigenvalues), and $R = R_{11} \oplus R_{22} \oplus R_{33} \in M_{2(q-w)}$, with

$$\begin{array}{rcl} R_{11} & = & \beta_1(-I_{q_1} \oplus I_{q_1}) \oplus \cdots \oplus \beta_{k_1}(-I_{q_{k_1}} \oplus I_{q_{k_1}}) \in M_{2(q-w)-u_1-u_2}, \\ R_{22} & = & \beta_{k_1+1}(-I_{q_{k_1+1}} \oplus I_{q_{k_1+1}}) \oplus \cdots \oplus \beta_{k_2}(-I_{q_{k_2}} \oplus I_{q_{k_2}}) \in M_{u_1}, \\ R_{33} & = & \beta_{k_2+1}(-I_{q_{k_2+1}} \oplus I_{q_{k_2+1}}) \oplus \cdots \oplus \beta_s(-I_{q_s} \oplus I_{q_s}) \in M_{u_2}, \end{array}$$

where $\beta_1, \ldots, \beta_s$ are pairwise distinct positive numbers such that for $i \leq k_1$, β_i is an eigenvalue of neither R_{44} nor R_{55}; for $k_1 < i \leq k_2$, β_i is an eigenvalue of R_{44} but not of R_{55}, and for $i > k_2$, β_i is an eigenvalue of R_{55} but not of R_{44}. Here, $q_i = \min\{m_1(\beta_i), m_2(\beta_i)\}$, where $m_1(\beta_i)$ and $m_2(\beta_i)$ denote the multiplicities of β_i in D_1 and D_2, respectively. Note that $w = 0$ implies $u_1 = u_2 = 0$.
Applying the same permutation similarity to A, we get

$$P^T A P = K \oplus I_w \oplus (-I_w),$$

where $K = K_{11} \oplus K_{22} \oplus K_{33}$, with

$$\begin{array}{rcl} K_{11} & = & [I_{q_1} \oplus (-I_{q_1})] \oplus \cdots \oplus \left[I_{q_{k_1}} \oplus (-I_{q_{k_1}})\right], \\ K_{22} & = & \left[I_{q_{k_1+1}} \oplus (-I_{q_{k_1+1}})\right] \oplus \cdots \oplus \left[I_{q_{k_2}} \oplus (-I_{q_{k_2}})\right], \\ K_{33} & = & \left[I_{q_{k_2+1}} \oplus (-I_{q_{k_2+1}})\right] \oplus \cdots \oplus [I_{q_s} \oplus (-I_{q_s})]. \end{array}$$

Then the equation $B = XAX$ is equivalent to

$$Y \begin{bmatrix} K & 0 & 0 \\ 0 & I_w & 0 \\ 0 & 0 & -I_w \end{bmatrix} Y = \begin{bmatrix} R & 0 & 0 \\ 0 & -R_{44} & 0 \\ 0 & 0 & R_{55} \end{bmatrix}, \tag{3.16}$$

where $Y = P^T X P$. Because of (3.14), Y has the form

$$Y = \left[\begin{array}{ccc|cc} Y_{11} & 0 & 0 & 0 & 0 \\ 0 & Y_{22} & 0 & Y_{24} & 0 \\ 0 & 0 & Y_{33} & 0 & Y_{35} \\ \hline 0 & Y_{24}^* & 0 & Y_{44} & 0 \\ 0 & 0 & Y_{35}^* & 0 & Y_{55} \end{array}\right], \tag{3.17}$$

where $Y_{11} \in M_{2(q-w)-u_1-u_2}$ is a direct sum of blocks of sizes $2q_1, \ldots, 2q_{k_1}$; $Y_{22} \in M_{u_1}$, $Y_{33} \in M_{u_2}$, and $Y_{44}, Y_{55} \in M_w$.
In particular, condition (3.16) implies

$$\begin{bmatrix} Y_{22} & Y_{24} \\ Y_{24}^* & Y_{44} \end{bmatrix} \begin{bmatrix} K_{22} & 0 \\ 0 & I_w \end{bmatrix} \begin{bmatrix} Y_{22} & Y_{24} \\ Y_{24}^* & Y_{44} \end{bmatrix} = \begin{bmatrix} R_{22} & 0 \\ 0 & -R_{44} \end{bmatrix}, \tag{3.18}$$

which is not possible for $w > 0$ because $K_{22} \oplus I_w$ and $R_{22} \oplus (-R_{44})$ have different inertia and, therefore, cannot be congruent. Thus, we deduce that $w = 0$, which implies $u_1 = u_2 = 0$. This means that D_1 and D_2 have the same eigenvalues. Because $YKY = R$, and taking into account the form of Y, it follows easily from Lemma 3.4 that Y, and therefore X, has q positive eigenvalues.

Conversely, suppose that D_1 and D_2 are similar. Then, there exists a permutation matrix $Q \in M_{2q}$ such that

$$Q^T \left[(-D_1) \oplus D_2\right] Q = T_1 \oplus \cdots \oplus T_s, \tag{3.19}$$

with $T_i = \beta_i \left(-I_{q_i} \oplus I_{q_i}\right), i = 1, \ldots, s$, where $\beta_1, \ldots, \beta_s$ are the distinct eigenvalues of D_1 (and D_2). According to Lemma 3.4, there is a real symmetric matrix $X_{2i} \in M_{2q_i}$, with q_i positive eigenvalues, such that $T_i = X_{2i}\left(I_{q_i} \oplus (-I_{q_i})\right) X_{2i}$. Then

$$X_2 \left[I_q \oplus (-I_q)\right] X_2 = (-D_1) \oplus D_2, \tag{3.20}$$

with $X_2 = Q(X_{21} \oplus \cdots \oplus X_{2s})Q^T$. Clearly, X_2 has q positive eigenvalues. □

We know describe the nonsingular simultaneously unitarily diagonalizable matrices $A, B \in M_n$ that are Hermitian-congruent.

Note that if $A, B \in M_n$ are two Hermitian matrices with the same inertia that are simultaneously unitarily similar to diagonal matrices, then AB is Hermitian and the number of negative eigenvalues of AB is even. Moreover, any unitary matrix that diagonalizes both A and B, also diagonalizes AB.

Theorem 3.6. *Let $A, B \in M_n$ be two nonsingular Hermitian matrices simultaneously unitarily similar to diagonal matrices. Let $2q$ be the number of negative eigenvalues of AB. When $q > 0$, let $u_1, \ldots, u_q$ be any orthonormal eigenvectors of A and B associated with positive eigenvalues of A and negative eigenvalues of B; let $u_{q+1}, \ldots, u_{2q}$ be any orthonormal eigenvectors of A and B associated with negative eigenvalues of A and positive eigenvalues of B. Then, there is a Hermitian matrix $X \in M_n$ with t positive eigenvalues such that $B = XAX$ if and only if $t \in \{q, \ldots, n-q\}$ and one of the following conditions is satisfied:*

1. *$q = 0$;*
2. *$q > 0$ and there is a permutation σ of $\{1, \ldots, q\}$ such that*

$$u_i^* AB u_i - u_{q+\sigma(i)}^* AB u_{q+\sigma(i)} = 0, \tag{3.21}$$

$i = 1, \ldots, q$.

Moreover, if A and B are real and $B = XAX$ has a Hermitian solution with t positive eigenvalues, then it has a real symmetric solution with t positive eigenvalues.

Proof. Suppose that A has $p + q$ positive eigenvalues. Let $v_1, \ldots, v_p$ be any orthonormal eigenvectors of A and B associated with positive eigenvalues of both A and B; let $w_1, \ldots, w_{n-p-2q}$ be any orthonormal eigenvectors of A and B associated with negative eigenvalues of both A and B. Then,

$$U = \begin{bmatrix} u_1 & \cdots & u_{2q} & v_1 & \cdots & v_p & w_1 & \cdots & w_{n-p-2q} \end{bmatrix}$$

is a unitary matrix such that

$$D_A = U^* A U = A_1 \oplus (-A_2) \oplus A_3 \oplus (-A_4), \tag{3.22}$$

$$D_B = U^* B U = (-B_1) \oplus B_2 \oplus B_3 \oplus (-B_4), \tag{3.23}$$

where $A_1, B_1, A_2, B_2 \in M_q$, $A_3, B_3 \in M_p$ and $A_4, B_4 \in M_{n-p-2q}$, are positive definite diagonal matrices. It is not hard to see that

$$u_i^* A u_i u_i^* B u_i = u_i^* A B u_i,$$

$i = 1, \ldots, 2q$. Thus, the eigenvalues of $A_1 B_1$ are

$$-u_1^* A B u_1, \ldots, -u_q^* A B u_q$$

and the eigenvalues of $A_2 B_2$ are

$$-u_{q+1}^* A B u_{q+1}, \ldots, -u_{2q}^* A B u_{2q}.$$

According to Lemma 2.4, A and B are Hermitian-congruent if and only if D_A and D_B are Hermitian-congruent.

Suppose that there is a Hermitian matrix $X \in M_n$ with t positive eigenvalues such that $B = XAX$. Then $D_B = Y D_A Y$, with $Y = U^* X U$. According to Lemma 3.2, $Y = Y_1 \oplus Y_2$, with $Y_1 \in M_{2q}$ and $Y_2 \in M_{n-2q}$ such that

$$Y_1 (A_1 \oplus (-A_2)) Y_1 = (-B_1) \oplus B_2 \tag{3.24}$$

and

$$Y_2 (A_3 \oplus (-A_4)) Y_2 = B_3 \oplus (-B_4). \tag{3.25}$$

If $q \neq 0$, according to Lemma 3.5, $A_1 B_1$ and $A_2 B_2$ are similar, which implies that there is a permutation σ of $\{1, \ldots, q\}$ such that

$$u_i^* A B u_i = u_{q+\sigma(i)}^* A B u_{q+\sigma(i)}, \tag{3.26}$$

for $i = 1, \ldots, q$. Also, Y_1 has q positive eigenvalues. Thus, $t \in \{q, \ldots, n-q\}$.

Conversely, suppose that $t \in \{q, \ldots, n-q\}$ and one of the conditions 1. or 2. is satisfied. By Lemma 3.3, there is a real symmetric matrix Y_2 with $t - q$ positive eigenvalues such that (3.25) holds. Since $A_1 B_1$ and $A_2 B_2$ are similar, by Lemma 3.5, there is a real symmetric matrix Y_1 with q positive eigenvalues such that (3.24) holds. Let $Y = Y_1 \oplus Y_2$. Then $B = XAX$, with $X = UYU^*$. Also, Y, and, therefore, X, has t positive eigenvalues. Clearly, in case A and B are real, the matrix U can be assumed to be real, which implies that X is real symmetric. □

The way we stated Theorem 3.6 was motivated by its analogy with Theorem 4.5. We now give an alternative characterization of the nonsingular Hermitian matrices, simultaneously unitarily similar to diagonal matrices, that are Hermitian-congruent (real symmetric congruent in the real case).

COROLLARY 3.7. *Let $A, B \in M_n$ be two nonsingular Hermitian matrices simultaneously unitarily similar to diagonal matrices. Let U be a unitary matrix such that $U^* A U = D_A$ and $U^* B U = D_B$, where D_A and D_B are diagonal matrices. Then, there is a Hermitian matrix $X \in M_n$ such that $B = XAX$ if and only if there is a permutation matrix $P \in M_n$ such that*

$$P^T D_A P = S_A \oplus A_1 \oplus \ldots \oplus A_q, \quad P^T D_B P = S_B \oplus B_1 \oplus \ldots \oplus B_q,$$

where

$$sign(S_A) = sign(S_B),$$

and $A_i, B_i \in M_2$ are indefinite matrices such that B_i is a negative multiple of A_i^{-1}, $i = 1, \ldots, q$.

4. The 2-by-2 case. In this section we study the existence of a Hermitian solution of $B = XAX$ when $A, B \in M_2$ are indefinite matrices. When A and B are real matrices, we show that $B = XAX$ has a Hermitian solution if and only if it has a real symmetric solution. We then show that if A and B are real matrices having the same sign pattern then A and B are congruent by a real symmetric matrix. In particular, if A and B are positive matrices, then they are congruent by a real symmetric matrix with arbitrary inertia.

4.1. General approach. We first consider the case in which $A = diag(1, -1)$ and B is real.

Lemma 4.1. *Let*

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \text{ and } B = \begin{bmatrix} a & t \\ t & b \end{bmatrix}, \tag{4.1}$$

with $a, b, t \in \mathbb{R}$ and $t \neq 0$. Then, there is a Hermitian matrix $X \in M_2$ such that $B = XAX$ if and only if there is a real z such that the following conditions are satisfied:

- $z^2 + a \geq 0$;
- $z^2 - b \geq 0$;
- $t = z(\sqrt{z^2 + a} + \varepsilon\sqrt{z^2 - b})$, *for some $\varepsilon \in \{-1, 1\}$.*

Moreover, if X is a Hermitian matrix such that $B = XAX$, then X is real.

Proof. Let

$$X = \begin{bmatrix} x & z \\ \overline{z} & y \end{bmatrix} \in M_2 \tag{4.2}$$

be a Hermitian matrix. Note that x and y are real numbers. Then $B = XAX$ is equivalent to

$$\begin{bmatrix} a & t \\ t & b \end{bmatrix} = \begin{bmatrix} x^2 - z\overline{z} & xz - zy \\ \overline{z}x - y\overline{z} & z\overline{z} - y^2 \end{bmatrix}, \tag{4.3}$$

which implies $xz - zy = \overline{z}x - y\overline{z} = t$. Since $t \neq 0$, then z must be real. Therefore, $B = XAX$ if and only if $z^2 + a \geq 0$, $z^2 - b \geq 0$,

$$x = \varepsilon_1\sqrt{z^2 + a}$$
$$y = -\varepsilon_2\sqrt{z^2 - b}$$

and

$$t = z(\varepsilon_1\sqrt{z^2 + a} + \varepsilon_2\sqrt{z^2 - b}), \tag{4.4}$$

for some $\varepsilon_1, \varepsilon_2 \in \{-1, 1\}$. Since the second member of (4.4) is an odd function of z, the result follows. □

We will need the following two technical lemmas.

Lemma 4.2. *Consider the function $f(z) = z(\sqrt{z^2 + c} + \sqrt{z^2 + d})$, with $c \geq d$, defined in $D_f = \{z \in \mathbb{R} : z^2 + d \geq 0\}$. Let $t \in \mathbb{R}$. Then there is $z \in \mathbb{R}$ such that $f(z) = t$ if and only if one of the following conditions is satisfied:*

1. $d \geq 0$;
2. $d < 0$ *and* $|t| \geq \sqrt{d(d - c)}$.

Proof. We have

$$\frac{df}{dz}(z) = \frac{(2z^2+c)\sqrt{z^2+d}+(2z^2+d)\sqrt{z^2+c}}{\sqrt{z^2+c}\sqrt{z^2+d}}, \tag{4.5}$$

$$\lim_{z\to-\infty} f(z) = -\infty \text{ and } \lim_{z\to+\infty} f(z) = +\infty. \tag{4.6}$$

Case 1: Suppose that $d \geq 0$. In this case f is a continuous function in $\mathbb{R}$ and, therefore, taking into account (4.6), $f(z) = t$ has a solution for any real t.

Case 2: Suppose that $d < 0$. Since f is continuous in D_f, $\frac{df}{dz}(z) \neq 0$ for any $z \in D_f$, $f(-\sqrt{-d}) = -\sqrt{d(d-c)}$ and $f(\sqrt{-d}) = \sqrt{d(d-c)}$, then, taking into account (4.6), $f(z) = t$ has a solution if and only if $|t| \geq \sqrt{d(d-c)}$. □

Lemma 4.3. *Consider the function* $g(z) = z(\sqrt{z^2+c} - \sqrt{z^2+d})$, *with* $c > d$ *and* $d \leq 0$, *defined in* $D_g = \{z \in \mathbb{R} : z^2 + d \geq 0\}$. *Let* $t \in \mathbb{R}$. *Then there is* $z \in \mathbb{R}$ *such that* $g(z) = t$ *if and only if one of the following conditions is satisfied:*

1. $c \leq -d$ *and* $|t| \in \left]\frac{c-d}{2}, \sqrt{d(d-c)}\right]$;
2. $c > -d$, $\sqrt{d(d-c)} \geq \frac{c-d}{2}$ *and* $|t| \in \left[\sqrt{-cd}, \sqrt{d(d-c)}\right]$;
3. $c > -d$, $\sqrt{d(d-c)} < \frac{c-d}{2}$ *and* $|t| \in \left[\sqrt{-cd}, \frac{c-d}{2}\right[$.

Proof. We have $g(-\sqrt{-d}) = -\sqrt{d(d-c)}$, $g(\sqrt{-d}) = \sqrt{d(d-c)}$,

$$\lim_{z\to-\infty} g(z) = -\frac{c-d}{2} \text{ and } \lim_{z\to+\infty} g(z) = \frac{c-d}{2}. \tag{4.7}$$

Also,

$$\frac{dg}{dz}(z) = \frac{(2z^2+c)\sqrt{z^2+d}-(2z^2+d)\sqrt{z^2+c}}{\sqrt{z^2+c}\sqrt{z^2+d}}. \tag{4.8}$$

Case 1: Suppose that $c \leq -d$. In this case $\frac{dg}{dz}$ has no roots. Since $g(\sqrt{-d}) > \frac{c-d}{2}$, then $g(z) = t$ has a solution if and only if condition 1. holds.

Case 2: Suppose that $c > -d$. Then $\frac{dg}{dz}$ has two roots: $z_1 = -\sqrt{-\frac{cd}{c+d}}$ and $z_2 = \sqrt{-\frac{cd}{c+d}}$. We have $g(z_1) = -\sqrt{-cd}$ and $g(z_2) = \sqrt{-cd}$. Since g is continuous in D_g, then $g(z) = t$ has a solution if and only if either condition 2. or condition 3. holds. □

We now use Lemmas 4.2 and 4.3 to obtain the following consequence of Lemma 4.1.

Lemma 4.4. *Let* $A = diag(1, -1)$, *and*

$$B = \begin{bmatrix} a & t \\ t & b \end{bmatrix} \tag{4.9}$$

be an indefinite real matrix with $t \neq 0$. *Then, there is a Hermitian matrix* X *such that* $B = XAX$ *if and only if one of the following conditions is satisfied:*

1. $a \geq b$;
2. $|t| > \frac{|a+b|}{2}$.

Proof. Note that, since B is indefinite, then $t^2 > ab$. According to Lemma 4.1, there is a Hermitian matrix X of the form (4.2) such that

$$\begin{bmatrix} a & t \\ t & b \end{bmatrix} = X \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} X, \tag{4.10}$$

if and only if there is $\varepsilon \in \{-1, 1\}$ such that the equation

$$t = z(\sqrt{z^2 + a} + \varepsilon\sqrt{z^2 - b}) \tag{4.11}$$

has a solution z. We now determine necessary and sufficient conditions for the existence of such an ε. Let $f(z) = z(\sqrt{z^2 + a} + \sqrt{z^2 - b})$ and $g(z) = z(\sqrt{z^2 + a} - \sqrt{z^2 - b})$. We first assume that $a \geq -b$.

Case 1: Suppose that $a \geq 0$.

Subcase 1.1: Suppose that $b \leq 0$. It follows from Lemma 4.2 that the equation $f(z) = t$ has a solution for any $t \in \mathbb{R}$.

Subcase 1.2: Suppose that $b > 0$.

- Suppose that $a > b$. Then, from Lemma 4.2, equation $f(z) = t$ has a solution for $|t| \in [\sqrt{b(b+a)}, +\infty[$; from Lemma 4.3, equation $g(z) = t$ has a solution for $|t| \in [\sqrt{ab}, \sqrt{b(b+a)}]$. Thus, for any t such that $t^2 > ab$, there is ε such that (4.11) has a solution.
- Suppose that $a \leq b$. Then, from Lemma 4.2, equation $f(z) = t$ has a solution if and only if $|t| \in [\sqrt{b(b+a)}, +\infty[$ and, from Lemma 4.3, equation $g(z) = t$ has a solution if and only if $|t| \in]\frac{a+b}{2}, \sqrt{b(b+a)}]$. Thus, there is ε such that (4.11) has a solution if and only if

$$|t| > \frac{a+b}{2}. \tag{4.12}$$

Case 2: Suppose that $a < 0$ and $b \geq 0$. From Lemmas 4.2 and 4.3, there is ε such that (4.11) has a solution if and only if (4.12) holds.

We showed that if $a \geq -b$ there is $\varepsilon \in \{-1, 1\}$ such that (4.11) has a solution if and only if

$$a \geq b \qquad \text{or} \qquad |t| > \frac{a+b}{2}. \tag{4.13}$$

As $g(-z) = -g(z)$, equation $g(z) = t$ has a solution if and only if equation $-g(z) = t$ has a solution. Therefore, for $\varepsilon \in \{-1, 1\}$, (4.11) has a solution if and only if

$$t = z(\sqrt{z^2 - b} + \varepsilon\sqrt{z^2 + a})$$

has a solution. Thus, if $a < -b$, by changing the roles of a and $-b$ in (4.13), it follows that there is ε such that (4.11) has a solution if and only if $a \geq b$ or $|t| > -\frac{a+b}{2}$. Then, the claim follows. $\square$

We now give the main result of this section. We consider that $A, B \in M_2$ are not simultaneously unitarily diagonalizable matrices, as the other case follows from Theorem 3.6.

THEOREM 4.5. *Let $A, B \in M_2$ be two indefinite matrices. Let u_1 and u_2 be orthonormal eigenvectors of A. Suppose that $u_1^* B u_2 \neq 0$. Then, there is a Hermitian matrix $X \in M_2$ such that $B = XAX$ if and only if one of the following conditions is satisfied:*

1. $u_1^*ABu_1 + u_2^*ABu_2 \geq 0$;

2. $|\sqrt{-u_1^*Au_1u_2^*Au_2}u_1^*Bu_2| > \frac{1}{2}|u_1^*ABu_1 - u_2^*ABu_2|$.

Moreover, if A and B are real and $B = XAX$ has a Hermitian solution, then it has a real symmetric solution.

Proof. Since the statement of the theorem is the same if we change the roles of u_1 and u_2, assume, without loss of generality, that u_1 is an eigenvector of A associated with the eigenvalue $\lambda_1 > 0$ and u_2 is an eigenvector of A associated with the eigenvalue $\lambda_2 < 0$. Moreover, by a possible multiplication of u_1 and u_2 by unit modulus complex numbers, assume that if A is real then u_1 and u_2 are real. Let $U = \left[\begin{array}{cc} u_1 & u_2 \end{array}\right]$. Then $U^*AU = diag(\lambda_1, \lambda_2)$. Let $D = diag(1/\sqrt{\lambda_1}, 1/\sqrt{-\lambda_2})$ and

$$B' = \left(D^{-1}U^*\right) B \left(UD^{-1}\right) = \left[\begin{array}{cc} a & te^{i\gamma} \\ te^{-i\gamma} & b \end{array}\right],$$

in which a and b are real (since B' is Hermitian), and t is real (in fact, we can even assume $t \geq 0$). Let $V = diag(e^{i\gamma}, 1)$. Then, for $C = UDV$, $C^*AC = diag(1, -1)$ and

$$C^{-1}BC^{-*} = \left[\begin{array}{cc} a & t \\ t & b \end{array}\right] \tag{4.14}$$

is real. Also, if A and B are real, then C is real. Note that

$$\begin{array}{rcl} a & = & \lambda_1 u_1^*Bu_1 \\ b & = & -\lambda_2 u_2^*Bu_2 \\ |t| & = & \sqrt{-\lambda_1\lambda_2}|u_1^*Bu_2| \end{array}$$

Because $B = XAX$ if and only if

$$\left[\begin{array}{cc} a & t \\ t & b \end{array}\right] = (C^{-1}XC^{-*}) \left[\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array}\right] (C^{-1}XC^{-*}), \tag{4.15}$$

conditions 1. and 2. follow from Lemma 4.4. If A and B are real and (4.15) holds, then, by Lemma 4.1, $C^{-1}XC$, and, therefore, X, is real. □

We finish by noting that if $A, B \in M_2$ are simultaneously unitarily diagonalizable matrices, then the conditions in Theorem 3.6 are equivalent to conditions 1. and 2. in Theorem 4.5, if in condition 2 we replace $>$ by $=$. Note that in this case $u_1^*Bu_2 = 0$.

4.2. A and B have the same sign pattern.

In this section we consider indefinite matrices $A, B \in M_2(\mathbb{R})$ and we study the existence of Hermitian solutions of $B = XAX$ in terms of the sign patterns of A and B. Note that according to Theorem 4.5, $B = XAX$ has a Hermitian solution if and only if it has a real symmetric solution. It turns out that if A and B have the same sign pattern, then there is always a Hermitian (real symmetric) solution to $B = XAX$, which is a remarkable result. In fact, if A and B are both positive, then there is always a positive definite solution, which by Theorem 2.3, implies that solutions with all possible inertias can be got.

The approach we follow here is based on Theorem 2.1.

Let $A, B \in M_2(\mathbb{R})$ be two indefinite matrices. Let $U, V \in M_2(\mathbb{R})$ be orthogonal matrices such that

$$A = UD_AU^t, \quad B = VD_BV^t, \tag{4.16}$$

where D_A and D_B are diagonal matrices.

It is easy to see that

$$U = \begin{bmatrix} cos(\theta) & sin(\theta) \\ sin(\theta) & -cos(\theta) \end{bmatrix}, \quad V = \begin{bmatrix} cos(\tau) & sin(\tau) \\ sin(\tau) & -cos(\tau) \end{bmatrix} \tag{4.17}$$

for some angles θ, τ. Also, we can assume, without loss of generality, that

$$D_A = \begin{bmatrix} l_1^2 & 0 \\ 0 & -l_2^2 \end{bmatrix}, \quad D_B = \begin{bmatrix} m_1^2 & 0 \\ 0 & -m_2^2 \end{bmatrix} \tag{4.18}$$

for some positive real numbers l_1, l_2, m_1, m_2 . Then we have

$$A = \begin{bmatrix} l_1^2 cos^2(\theta) - l_2^2 sin^2(\theta) & (l_1^2 + l_2^2) sin(\theta) cos(\theta) \\ (l_1^2 + l_2^2) sin(\theta) cos(\theta) & l_1^2 sin^2(\theta) - l_2^2 cos^2(\theta) \end{bmatrix}, \tag{4.19}$$

$$B = \begin{bmatrix} m_1^2 cos^2(\tau) - m_2^2 sin^2(\tau) & (m_1^2 + m_2^2) sin(\tau) cos(\tau) \\ (m_1^2 + m_2^2) sin(\tau) cos(\tau) & m_1^2 sin^2(\tau) - m_2^2 cos^2(\tau) \end{bmatrix}. \tag{4.20}$$

Let Y and Z be nonsingular matrices satisfying $A = Y^T K Y$ and $B = Z^T K Z$, with $K = diag(1, -1)$. It follows from Theorem 2.1, that A and B are congruent by a real symmetric matrix if and only if there exists a real K-unitary matrix W such that WZY^t is symmetric. Moreover, $X \in M_2(\mathbb{R})$ is a symmetric solution of $B = XAX$ if and only if $X = Y^{-1} W Z$ for some real K-unitary matrix W such that WZY^T is symmetric. According to [2], any real K-unitary matrix W can be written as

$$W_1 = \begin{bmatrix} sec(\alpha) & tan(\alpha) \\ tan(\alpha) & sec(\alpha) \end{bmatrix}, \quad \text{or} \quad W_2 = \begin{bmatrix} -sec(\alpha) & -tan(\alpha) \\ tan(\alpha) & sec(\alpha) \end{bmatrix}, \tag{4.21}$$

for some angle α. Taking into account the two possible forms for W, we introduce the following definition.

DEFINITION 4.6. *Let $A, B \in M_2(\mathbb{R})$ be two indefinite matrices. Let $A = Y^t K Y$ and $B = Z^t K Z$, where $K = diag(1, -1)$. If there exists a real K-unitary matrix W such that WZY^t is real symmetric, then we say that $X = Y^{-1} W Z$ is*

- *a solution of type 1 of $B = XAX$ if W has the form of W_1 in (4.21).*
- *a solution of type 2 of $B = XAX$ if W has the form of W_2 in (4.21).*

Then, we have the following result.

LEMMA 4.7. *Let $A, B \in M_2(\mathbb{R})$ be indefinite matrices. Then, A and B are Hermitian-congruent if and only if there exists a real number α such that $cos(\alpha) \neq 0$ and one of the following conditions holds:*

1. *$(m_1 l_2 + l_1 m_2) sin(\theta - \tau) = (m_1 l_1 - l_2 m_2) sin(\alpha) cos(\theta - \tau)$, in which case there is a solution of type 1;*
2. *$(m_2 l_1 - l_2 m_1) sin(\theta - \tau) = (m_1 l_1 + l_2 m_2) sin(\alpha) cos(\theta - \tau)$, in which case there is a solution of type 2.*

Proof. Noting that

$$R = ZY^T = \begin{bmatrix} l_1 m_1 cos(\theta - \tau) & l_2 m_1 sin(\theta - \tau) \\ -l_1 m_2 sin(\theta - \tau) & l_2 m_2 cos(\theta - \tau) \end{bmatrix}.$$

and considering the expressions for W_1 and W_2 given in (4.21), the result follows in a straightforward way by checking when WR is symmetric. □

It is worth to remark that the results in this subsection are independent of the selection of the matrices Z and Y.

We now study the existence of a real symmetric solution to $B = XAX$ in terms of the sign patterns of A and B. Here we consider sign patterns with no zero entries. Thus, the following are the only possibilities:

$$P_1 = \begin{bmatrix} + & + \\ + & + \end{bmatrix}, \quad P_2 = \begin{bmatrix} + & + \\ + & - \end{bmatrix}, \quad P_3 = \begin{bmatrix} - & + \\ + & + \end{bmatrix}, \quad P_4 = \begin{bmatrix} - & + \\ + & - \end{bmatrix},$$

$$P_5 = \begin{bmatrix} - & - \\ - & - \end{bmatrix}, \quad P_6 = \begin{bmatrix} - & - \\ - & + \end{bmatrix}, \quad P_7 = \begin{bmatrix} + & - \\ - & - \end{bmatrix}, \quad P_8 = \begin{bmatrix} + & - \\ - & + \end{bmatrix}. \tag{4.22}$$

The main result in this subsection shows that if A and B are real indefinite matrices having the same sign pattern, then there is always a real symmetric solution of $B = XAX$. Notice that in order to prove that, it is enough to check that the result is true for the patterns P_1 and P_2 as

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} P_3 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = P_2, \tag{4.23}$$

$$P_4^{-1} = P_1, \quad P_5 = -P_1, \quad P_6 = -P_2, \quad P_7 = -P_3, \quad P_8 = -P_4.$$

The following lemma will be useful in the proofs of our results.

LEMMA 4.8. *Consider the function* $f(x,y) = \frac{x-y}{1+xy}$ *defined in* $S = [a_1\ a_2] \times [b_1\ b_2]$, *with* $0 \le a_1 < a_2$ *and* $0 \le b_1 < b_2$. *Then,*

$$f(a_1, b_2) \le f(x,y) \le f(a_2, b_1) \quad \text{for all } (x,y) \in S.$$

Proof. The function $f(x,y)$ is differentiable and it is defined in a compact set. It achieves the absolute maximum and the absolute minimum on the boundary of S since

$$\frac{\delta f}{\delta x} = \frac{1+y^2}{(1+xy)^2} > 0, \quad \frac{\delta f}{\delta y} = \frac{-1-x^2}{(1+xy)^2} < 0 \quad \text{for all } (x,y) \in S.$$

Note that the domain of f is a subset of $\mathbb{R}^2$ with rectangular form, and the partial derivatives of f have also constant sign in each of the edges of this rectangle. Taking into account that $f(x, b_1)$ and $f(x, b_2)$ are both increasing functions in the interval $[a_1, a_2]$, while $f(a_1, y)$ and $f(a_2, y)$ are decreasing functions in the interval $[b_1, b_2]$, the result follows. □

4.2.1. A **and** B **are positive matrices.** It is well known [3] that if M is a positive matrix, then the spectral radius $\rho(M)$ of M is positive, $\rho(M)$ is an eigenvalue of M and there is a positive eigenvector associated with $\rho(M)$. Moreover, if λ is any other eigenvalue of M, then $|\lambda| < \rho(M)$.

Let $A, B \in M_2(\mathbb{R})$ be the matrices given in (4.19) and (4.20), respectively, and assume that A and B are positive. Then,

$$l_2 < l_1, \quad m_2 < m_1. \tag{4.24}$$

$$\cos(\theta) \neq 0, \quad \cos(\tau) \neq 0, \quad \tan(\theta) > 0, \quad \tan(\tau) > 0, \tag{4.25}$$

and

$$\frac{l_2}{l_1} < \tan(\theta) < \frac{l_1}{l_2}, \quad \frac{m_2}{m_1} < \tan(\tau) < \frac{m_1}{m_2}. \tag{4.26}$$

Notice that if the equation $B = XAX$ has a Hermitian solution, then $cos(\theta - \tau)$ must be different from zero. If $cos(\theta - \tau) = 0$, then $\theta - \tau = (2r+1)\pi/2$ for $r = 0, 1, 2, ...$ But this implies that $tan(\theta)tan(\tau) < 0$, which contradicts (4.25).

Next we analyze when there exist solutions of type 1 or type 2 to $B = XAX$.

Solutions of type 1. It follows from Lemma 4.7 that a solution of type 1 of $B = XAX$ exists if and only if

$$-1 < C_1 \tan(\theta - \tau) < 1, \tag{4.27}$$

where

$$C_1 = \frac{m_1 l_2 + l_1 m_2}{m_1 l_1 - l_2 m_2} > 0. \tag{4.28}$$

Moreover, if a solution of type 1 exists, then there are exactly two solutions of type 1, say X_1 and X_2. By Lemma 4.7, one corresponds to $\alpha = arc\sin(C_1 \tan(\theta - \tau))$ and the other corresponds to $\alpha = arc\sin(C_1 \tan(\theta - \tau)) + \pi$, which implies that $X_1 = -X_2$. Taking into account that

$$tan(\theta - \tau) = \frac{tan(\theta) - tan(\tau)}{1 + tan(\theta)tan(\tau)},$$

it follows easily from Lemma 4.8 and (4.26), that

$$-\frac{1}{C_1} < \tan(\theta - \tau) < \frac{1}{C_1}. \tag{4.29}$$

Thus, (4.27) holds and we have the following result.

THEOREM 4.9. *Let $A, B \in M_2(\mathbb{R})$ be indefinite positive matrices. Then the equation $B = XAX$ has two solutions of type* 1.

Solutions of type 2. It follows from Lemma 4.7 that a solution of type 2 of $B = XAX$ exists if and only if

$$-1 < C_2 \tan(\theta - \tau) < 1, \tag{4.30}$$

where

$$C_2 = \frac{m_2 l_1 - l_2 m_1}{m_1 l_1 + l_2 m_2}. \tag{4.31}$$

Again, if a solution X of type 2 exists, then there are exactly two solutions of type 2, X and $-X$.

THEOREM 4.10. *Let $A, B \in M_2(\mathbb{R})$ be indefinite positive matrices. Then the equation $B = XAX$ has two solutions of type* 2.

Proof. Taking into account that $m_1 l_1 > m_2 l_2$ and $m_2 < m_1$, it is easy to prove that $C_2/C_1 < 1$. Considering (4.27), it follows that there is a solution X of type 2. Then $-X$ is also a solution of type 2.
$\square$

Definite solutions. In this part we determine the existence of definite real symmetric solutions to $B = XAX$ when $A, B \in M_2(\mathbb{R})$ are indefinite positive matrices. Note that in general we may have a Hermitian solution of $B = XAX$ but not a definite solution. For example, the equation $diag(1, -1) = X(-1, 1)X$ has a Hermitian solution but has no definite solutions (see Lemma 3.5).

By Theorem 4.9, there exist two solutions of type 1 of $B = XAX$, say X and $-X$, with

$$X = Y^{-1}WZ = U|D_A|^{-1/2}W|D_B|^{1/2}V^t, \tag{4.32}$$

where

$$W = \frac{1}{cos(\alpha)} \begin{bmatrix} 1 & C_1 tan(\theta - \tau) \\ C_1 tan(\theta - \tau) & 1 \end{bmatrix}, \quad \alpha = arcsin(C_1 tan(\theta - \tau)).$$

We show here that these solutions are definite matrices. Then, taking into account Theorem 2.3, we can assure that the solutions of type 2 are both indefinite. A calculation shows that

$$U^t XU = \frac{1}{cos(\alpha)} \begin{bmatrix} \frac{m_1}{l_1} cos(\theta - \tau) - \frac{m_2}{l_1} C_1 \frac{sin^2(\theta-\tau)}{cos(\theta-\tau)} & \left(\frac{m_1}{l_1} + \frac{m_2 C_1}{l_1}\right) sin(\theta - \tau) \\ \left(\frac{m_1 C_1}{l_2} - \frac{m_2}{l_2}\right) sin(\theta - \tau) & \frac{m_1}{l_2} C_1 \frac{sin^2(\theta-\tau)}{cos(\theta-\tau)} + \frac{m_2}{l_2} cos(\theta - \tau) \end{bmatrix}.$$

It is easy to check that this is a symmetric matrix, which implies that X also is. Using (4.29), it is also easy to see that $det(U^T XU) > 0$. Thus, $U^T XU$, and, consequently, X, is definite. Therefore, we have shown that the two solutions X and $-X$ of type 1 are definite, and we get the following result.

THEOREM 4.11. *Let $A, B \in M_2(\mathbb{R})$ be indefinite positive matrices. Then the equation $B = XAX$ has two definite real solutions as well as two indefinite real solutions.*

The results obtained for positive matrices are inherited by 2-by-2 real indefinite matrices with sign pattern P_4.

COROLLARY 4.12. *Let $A, B \in M_2(\mathbb{R})$ be indefinite matrices with sign pattern P_4. Then the equation $B = XAX$ has two solutions of type 1 and two solutions of type 2. Moreover, one of the solutions is positive definite.*

Proof. Notice that if a real indefinite matrix M has sign pattern P_4, then M^{-1} is a positive matrix. Notice also that $B = XAX$ if and only if $B^{-1} = X^{-1}A^{-1}X^{-1}$. Taking into account that X is positive definite if and only if X^{-1} is positive definite, the result follows from Theorems 4.9, 4.10, and 4.11. □

It is trivial to note that the same results are also inherited by matrices A and B having both sign pattern either P_5 or P_8.

4.2.2. A and B have sign pattern P_2.

Let $A, B \in M_2(\mathbb{R})$ be the matrices given in (4.19) and (4.20), respectively, and assume that A and B have sign pattern P_2. Then, $\cos(\theta) \neq 0$, $\cos(\tau) \neq 0$ and

$$0 < \tan(\theta) < \min\left\{\frac{l_1}{l_2}, \frac{l_2}{l_1}\right\}, \quad 0 < \tan(\tau) < \min\left\{\frac{m_1}{m_2}, \frac{m_2}{m_1}\right\}. \tag{4.33}$$

Notice that, using the same argument as for positive matrices, it can be proven that $\cos(\theta - \tau) \neq 0$.

Theorem 4.13. *Let* $A, B \in M_2(\mathbb{R})$ *be indefinite matrices with sign pattern* P_2. *Then the equation* $B = XAX$ *has two solutions of type* 2.

Proof. Since $cos(\theta - \tau) \neq 0$, the equation $B = XAX$ has a solution of type 2 (in fact, two solutions) if and only if

$$-1 < C_2\, tan(\theta - \tau) < 1, \tag{4.34}$$

where C_2 is as in (4.31). Using Proposition 4.8 and (4.33), we have

$$-\min\left\{\frac{m_1}{m_2}, \frac{m_2}{m_1}\right\} \leq tan(\theta - \tau) \leq \min\left\{\frac{l_1}{l_2}, \frac{l_2}{l_1}\right\} \tag{4.35}$$

It is easy to see that

$$C_2 < \min\left\{\frac{m_2}{m_1}, \frac{l_1}{l_2}\right\}, \text{ if } C_2 > 0 \tag{4.36}$$

and

$$-C_2 < \min\left\{\frac{m_1}{m_2}, \frac{l_2}{l_1}\right\}, \text{ if } C_2 < 0. \tag{4.37}$$

Thus, multiplying (4.35) by C_2 and using (4.36) and (4.37), condition (4.34) follows. Note that

$$\min\left\{\frac{l_1}{l_2}, \frac{l_2}{l_1}\right\} \min\left\{\frac{m_2}{m_1}, \frac{l_1}{l_2}\right\} \leq 1; \qquad \min\left\{\frac{m_1}{m_2}, \frac{m_2}{m_1}\right\} \min\left\{\frac{m_2}{m_1}, \frac{l_1}{l_2}\right\} \leq 1$$
$$\min\left\{\frac{l_1}{l_2}, \frac{l_2}{l_1}\right\} \min\left\{\frac{m_1}{m_2}, \frac{l_2}{l_1}\right\} \leq 1; \qquad \min\left\{\frac{m_1}{m_2}, \frac{m_2}{m_1}\right\} \min\left\{\frac{m_1}{m_2}, \frac{l_2}{l_1}\right\} \leq 1.$$

□

Corollary 4.14. *Let* $A, B \in M_2(\mathbb{R})$ *be indefinite matrices with sign pattern* P_3*,* P_6 *or* P_7. *Then the equation* $B = XAX$ *has two solutions of type* 2.

The next example shows that when A and B are real indefinite matrices with sign pattern P_2, the equation $B = XAX$ may have no solutions of type 1.

Consider A and B as in (4.19) and (4.20), respectively, with $l_1 = l_2 = 1$, $m_1 = 1.6$, $m_2 = 1.4$, $\theta = \pi/6$ and $\tau = \pi/12$. It is easy to see that A and B are real indefinite matrices with sign pattern P_2. Also, $C_1 \tan(\theta - \tau) = 3.642155906 > 1$. Thus, a solution of type 1 does not exist. On the other hand, if X is a solution given by (4.32), where W has the form of W_2 in (4.21), then $\det(X) < 0$, which implies that X is indefinite. Thus, $B = XAX$ has no definite solutions.

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