GARSIDE STRUCTURES AS COMBINATORIAL OBJECTS

NOEL BRADY [1], JOHN CRISP, ANTON KAUL, AND JON MCCAMMOND [2]

Abstract. Every Garside group contains a poset of divisors with several unusual properties. In this article we attempt to characterize the labeled posets that arise in this way. Towards this end we define a class of objects we call combinatorial Garside structures. Each of these is a labeled poset with properties sufficient to guarantee the existence of a Garside group containing it. Although our characterization is not quite complete (because not all Garside groups are known to arise in this fashion) every Garside group that has so far occured in the literature can be constructed using this procedure.

Along the way, we highlight how two properties usually presumed (namely, that the poset of divisors is both finite and a lattice) can be weakened considerably while still producing groups that behave like Garside groups in most important respects. This extension of the theory to infinite posets that are not quite lattices was prompted by our investigation of the Garside-like structures contained in arbitrary Artin groups [4, 5] and represent the core of the new results being presented here.

The study of Garside groups has flourished recently within geometric group theory both as a way to better understand Artin groups of finite-type and as an interesting class of groups in their own right [1, 2, 3, 6, 7, 10, 11, 13]. Roughly speaking, a Garside group, in the classical sense, is a group that has a finite generating set equipped with a lattice structure. In this article we investigate how to characterize the finite lattices that arise in this fashion. Although we are not able to give a complete characterization, we are at least able to characterize those lattices associated with Garside groups that satisfy an additional minor condition which we call being weakly graded. Our first main theorem is the following.

Theorem A (Equivalence). *There is a natural bijection between combinatorial Garside structures and weakly graded Garside groups.*

That such a theorem exists will not come as a surprise to the experts in the area. Our main goal, in fact, lies elsewhere. We wish to significantly enlarge the class of combinatorial Garside structures without jeopardizing the usual consequences which follow from the existence of a Garside structure. Recall that to every labeled poset P there is a naturally defined monoid M, group G and complex K derived from P [15]. These constructions are briefly reviewed during the course of the article. Combinatorial Garside structures and their associated Garside groups are of particular interest because of the close connections among P, M, G and K. The following is a partial list of these connections.

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Theorem B (Consequences)**.** *If P is a combinatorial Garside structure and M, G and K are the monoid, group and complex derived from P then*

- $\textsc{Hasse}(P, S)$ *embeds into* $\textsc{Cayley}(M, S)$,
- $\textsc{Cayley}(M, S)$ *embeds into* $\textsc{Cayley}(G, S)$,
- *G is the group of fractions of M,*
- *the word problem for G is solvable, and*
- *the universal cover of K is contractible.*

This final property makes K an Eilenberg-MacLane space for G. It thus follows that the cohomology of G is equal to that of K, the cohomological dimension of G is bounded above by the dimension of K which in turn is equal to the height of P. It also follows that the group G is torsion-free.

Additional consequences certainly exist—such as the existence of biautomatic structures [10] and a regular language of geodesics [8]— but since these do not generalize as readily to infinite posets, they play only a secondary role in this article. The standard proofs of these consequences are reviewed here in order to facilitate their extension first to infinite posets, and then to posets that are not quite lattices. More specifically, the proofs we present have been recast so that they apply immediately to what could be called *infinite Garside structures.* This sometimes requires a slight reworking of previously implicit hypotheses, but it is otherwise straightforward. As an example, the assertion that the word problem for G is solvable is more carefully stated as follows.

Theorem C (Word problem)**.** *Let P be a possibly infinite combinatorial Garside structure and let G be the group derived from P. If the elements of P can recursively described, equality in P can be algorithmically tested and meets and joins of elements in P can be algorithmically produced, then the word problem for G is solvable.*

When P is finite, these hypotheses are trivially satisfied but they needed to be made explicit in the infinite version. The extension to posets which are not lattices represents a more radical departure. Every partially order set can be embedded into a complete lattice in a minimal, canonical way. What we have found is that so long as the inclusion of the original poset into its completion is sufficiently well-behaved, the main consequences, such as those listed in Theorem B, can still be established. The idea is to pass to the completion whenever the lattice property is needed and then to argue that the needed properties descend back down to the original poset. See §5 for details.

Overview: The structure of the article is as follows. The traditional notion of a Garside group is quickly reviewed in §1 and combinatorial Garside structures are introduced in §2, where Theorem A is established. In §3 we prove a version of Theorem B that does not presume the finiteness of P. Finally, after reviewing the lattice completion of a poset in §4, all of our main results are extended in §5 to the case where P is merely "nearly a lattice" in the precise technical sense we describe.

1. Garside monoids and Garside groups

This section reviews the definition of a Garside monoid and a Garside group. The central role played by the poset of divisors of the Garside element and the properties this poset possesses are particularly highlighted.

Definition 1.1 (Atomic monoid)**.** Let M be a monoid. If $m = m_1 \cdot m_2 \cdots m_n$ in M with all m_i nontrivial then m is said to have a *nontrivial factorization of length* n. The supremum of the lengths of all nontrivial factorizations of m is called its *norm*. Since the norm of the identity element is conventionally defined as 0, this defines a map $|| \cdot || : M \to \mathbb{N} \cup \{\infty\}$ with the property that $||m|| > 0$ for all non-indentity elements and $||mm'|| \geq ||m|| + ||m'||$ for all for all m and m' in M. If the norm satisfies the stronger property that $||mm'|| = ||m|| + ||m'||$ then M is *graded*. The *indivisibile* elements in M, i.e. the nontrivial elements of M with no nontrivial factorizations, are also called its *atoms* and they are precisely the elements of norm 1. A monoid is *atomic* if every element of M has a finite norm. The name alludes to the fact that atomic monoids are necessarily generated by their atoms.

Certainly not all monoids are atomic. The nonnegative rationals under addition have no atoms at all, and even when a monoid is finitely generated, any element that has a nontrivial left or right stabilizer will have an infinite norm. To each atomic monoid, there are naturally defined posets.

Definition 1.2 (Divisibility)**.** If $ab = c$ in a monoid M then a is a *left divisor* of c, b is a *right divisor* of c, and c is a *right multiple* of a and a *left multiple* of b. In symbols $a \leq_L c$ and $b \leq_R c$. When the unmentioned letter is nontrivial, the adjective *proper* can be added. For example, when b is nontrivial a is a *proper left divisor* of c. In any monoid, left divisibility is reflexive and transitive, but in an atomic monoid it is anti-symmetric as well. Thus $(M, \leq_L)$ is a partially ordered set called the *left division poset* of M and it makes sense to write $a <_L c$ when a is a proper left divisor of c. Similar statements hold for $(M, \leq_R)$, of course.

Garside monoids are atomic monoids where the left division and right division posets have certain additional properties.

Definition 1.3 (Garside monoid)**.** Let M be an atomic monoid, let Δ be an element of M and let $L(\Delta)$ and $R(\Delta)$ denote the set of left divisors of Δ and the set of right divisors of Δ, respectively. The monoid M is a *Garside monoid* with *Garside element* Δ if

(G1) M is left and right cancellative,
(G2) the posets $(M, \leq_L)$ and $(M, \leq_R)$ are lattices,
(G3) the sets $L(\Delta)$ and $R(\Delta)$ are equal,
(G4) the set $L(\Delta)$ generates M, and
(G5) the set $L(\Delta)$ is finite,

If M is an atomic monoid with element Δ where $L(\Delta)$ is infinite, but the other four conditions hold, then M is called a *Garside monoid with an infinite Garside structure* and Δ is its Garside element. Inside Garside monoids with finite or infinite Garside structures, the set $L(\Delta) = R(\Delta)$ is called the *divisors of* Δ and sometimes denoted $\mathcal{D}$.

The divisors of Δ clearly play a major role in the definition of a Garside structure. The main result of this section, which is fairly immediate once all the definitions have been given, is a description of their key properties.

Theorem 1.4 (Properties of divisors)**.** *If M is a Garside monoid with a possibly infinite Garside structure and Δ is its Garside element, then the divisors of Δ form a bounded, finite-height lattice with a balanced, group-like labeling.*

The remainder of the section introduces the terminology used in Theorem 1.4. Some of it is new, but much of it is standard combinatorial language. Throughout this section, let M denote a monoid with a possibly infinite Garside structure and let Δ denote its Garside element.

Definition 1.5 (Bounded)**.** Let $(P, \leq)$ be an arbitrary poset. If P contains a maximum element, i.e. an element p such that $p \geq q$ for all $q \in P$, then it is often denoted $\hat{1}$. Similarly, if there is a minimum element it is often denoted $\hat{0}$. If P contains both types of elements then P is *bounded.* For each subset $Q \subset P$ there is an induced *subposet* structure on Q which is simply the restriction of the order on P. Given any pair of elements $p \leq q$ in P, the subposet on the set $[p, q] := \{r \mid p \leq r \leq q\}$ is called an *interval* of P. The interval $[p, q]$ is, of course, bounded by p and q and these are called its *endpoints.* The collection $I(P) = \{(p, q) | p \leq q\}$ is called the *set of intervals in* P.

The (left) divisors of Δ naturally correspond to the elements in the interval $[1, \Delta]$ inside the poset $(M, \leq_L)$, where 1 denotes the identity element of M. This interval is called the *poset of left divisors* and it is this poset that possesses the properties listed in Theorem 1.4. Since it is defined as an interval, it is clearly bounded.

Definition 1.6 (Finite-height)**.** Two elements p, q are *comparable* if $p \leq q$ or $q \leq p$ and *incomparable* otherwise. A poset is *linear* (and the partial order is called a *total order*) if every pair of elements is comparable. A subposet C which is linear is called a *chain.* The *length* of a finite chain C is $|C| - 1$. Finite chains are bounded and its maximum and minimum elements are its *endpoints.* If a finite chain C is not a subposet of a strictly larger finite chain with the same endpoints, then C is *saturated.* A saturated chain of length 1 is called a *covering relation.* In a bounded poset, the supremum of the lengths of chains whose endpoints are $\hat{0}$ and $\hat{1}$ is called its *height* and the elements which cover $\hat{0}$ are called its *atoms.* Finally, if every saturated chain in P between the same pair of endpoints has the same finite length, then P is called a *graded poset.*

Definition 1.7 (Local properties)**.** If every interval in a poset has a particular property then P is said to *locally* have that property. For example, if every interval contains only finitely many elements, then P is a *locally finite* poset and if every interval in P has finite height then P *locally has finite height.*

In any poset with a minimum element $\hat{0}$, we can define a height function $h : P \to \mathbb{N} \cup \infty$ which sends p to the height of the interval $[\hat{0}, p]$. This number is the *height* or *rank* of p. The element $\hat{0}$ has rank 0 and atoms are the elements with rank 1. Notice that in a poset with a minimum element, it locally has finite height if and only if every element has finite height. Moreover, in a graded poset with a minimum element $\hat{0}$, the elements can be split up into levels based on their heights and every saturated chain connects elements in successive ranks.

Remark 1.8 (Monoid vs. Poset)**.** Comparing the poset and monoid definitions we find that a monoid M is atomic if and only if left divisibility in M defines a poset $(M, \leq_L)$ which locally has finite height. Moreover, the overlaps in notation are consistent. In an atomic monoid M, m is an atom in the monoid M if and only if m is an atom in the poset $(M, \leq_L)$. Similarly, M is a graded atomic monoid if and only if $(M, \leq_L)$ is a graded poset. Finally, the norm of an element of M is the same as its height or rank when viewed as an element of $(M, \leq_L)$.

Figure 1. The group-like condition involves pairs of 3 element chains such as the ones shown.

Returning to our proof of Theorem 1.4, since M is an atomic monoid, it follows that $(M, \leq_L)$ locally has finite height and, in particular, the poset of divisors has finite height.

Definition 1.9 (Lattice)**.** Let $(P, \leq)$ be a poset and let Q be some subset of P. An element $p \in P$ such that $p \geq q$ for all $q \in Q$ is called an *upper bound* for Q. If the set of upper bounds for Q has a minimum element, this element is called the *least upper bound* or *join* of Q and denoted $\bigvee Q$. The notions of a *lower bound* for Q and a *greatest lower bound* or *meet* for Q are defined similarly. The meet of Q is denoted $\bigwedge Q$. For two element sets such as $Q = \{p, q\}$ the join and meet are written $p \vee q$ and $p \wedge q$ respectively. If every pair of elements in P has a meet and a join then $(P, \leq)$ is called a *lattice*. If every subset Q has a meet and a join then $(P, \leq)$ is called a *complete lattice*.

It is easy to show that if P is a lattice, then every interval in P is also a lattice. Moreover, when a bounded lattice has finite height, it must be complete. As a consequence, the poset of divisors in M is not only a lattice, it is a complete lattice. The final set of properties all depend on the existence of a labeling.

Definition 1.10 (Labeling)**.** A *labeling* of a poset P, or more specifically an *interval labeling* of P, is any function $\lambda : I(P) \to S$ from the intervals of P to a set of *labels*. Typically we restrict S to the image of λ so that S is the set of labels actually used.

If M is a cancellative atomic monoid, then the poset $P = (M, \leq_L)$ is naturally labeled by the elements of M. In particular, if $m \leq_L m'$ then there exists at least one element $m'' \in M$ with $mm'' = m'$ and because of left cancellation, this element m'' is unique. Thus we can define $\lambda : I(P) \to M$ by setting $\lambda(m, m')$ equal to this well-defined m''. By restriction, the poset of divisors inherits a labeling by some subset of M. We shall see below that this set of labels is precisely the set of divisors. Since this labeling comes from a cancellative monoid, it is naturally group-like in the following sense.

Definition 1.11 (Group-like)**.** An interval-labeled poset P is called *group-like* if whenever two 3 element chains $x \leq y \leq z$ and $x' \leq y' \leq z'$ have two pairs of corrsponding labels in common, then the third pair of labels are also equal. See Figure 1. The three possible configurations of common labels ensure that the labeling is left cancellative, right cancellative and multiplicative.

The final property that needs to be defined is balanced. This is nothing more than a translation of axiom (G3) into combinatorial terms. In particular, the set of left divisors $L(\Delta)$ is equal to the labels on the intervals of the form $1 \leq_L p$ with $p \in [1, \Delta]$ and the set of right divisors $R(\Delta)$ is equal to the set of labels on the intervals of the form $p \leq_L \Delta$. This leads to the following general definition.

Definition 1.12 (Balanced)**.** Let $(P, \leq)$ be a bounded poset labeled by the function $\lambda : I(P) \to S$ and consider the follows sets of labels.

$$L(P) := \{ \ \lambda(\hat{0}, q) \ : \ q \in P\}$$

$$R(P) := \{ \ \lambda(p, \hat{1}) \ : \ p \in P\}$$

Figure 2. The type of diagram used to prove that left and right common multiples always exist.

If $L(P) = R(P)$ then the labeling on P is said to be *balanced.*

By axiom (G3) the labeling on the poset of divisors is balanced and this completes the proof of Theorem 1.4. In any cancellative atomic monoid M, call an element m a *balanced element* if the interval $[1, m]$ in $(M, \leq_L)$, naturally labeled by some subset of M, is balanced. Using this terminology, the Garside element Δ is a balanced element. As promised, we can now show that the set of labels used in the poset of divisors is precisely the set of divisors themselves.

Lemma 1.13 (Divisors as labels)**.** *Let M is a cancellative atomic monoid, let m be an element in M, and let P be the interval $[1, m]$ in $(M, \leq_L)$ naturally labeled by some subset S in M. If m is balanced, then $L(P) = R(P) = S$.*

Proof. Since balanced means $L(P) = R(P) \subseteq S$, we only need to show that for every $(p, q) \in I(P)$, the label $p, q)$ lies in $L(P)$. If we let $a = \lambda(1, p)$, $b = \lambda(p, q)$ and $c = \lambda(q, m)$, then $abc = m$, and bc is a right divisor of m. Because m is balanced, bc is also a left divisor of m and $m = bca'$ for some $a' \in M$. Thus b is a left divisor of m and $\lambda(p, q)$ is in $L(P)$. □

Notice that this result does not need either a lattice structure for $(M, \leq_L)$ or a finite number of atoms. We conclude this section by introducing Garside group in a way that similarly avoids using either of these two assumptions.

Definition 1.14 (Fundamental element)**.** Let M be a cancellative atomic monoid. In the left division poset $(M, \leq_L)$, an upper bound of a subset is sometimes called a *right common multiple* and the join, if it exists, is called a *right least common multiple* or *right lcm.* If a balanced element $m \in M$ is a right common multiple of the atoms of M, then m is called a *fundamental element.*

Garside elements are certainly examples of fundamental elements, but the number of atoms need not be finite and the division orders need not be lattices. Fundamental elements are important because they guarantee the existence of left and right common multiples for all pairs of elements.

Lemma 1.15 (Right common multiples)**.** *If M is a cancellative atomic monoid with a fundamental element m, then every pair of elements in M have a right common multiple and a left common multiple. As a consequence, M embeds into its group of fractions.*

Proof. The proof is essentially contained in the diagram shown in Figure 2. Let P be the labeled poset $[1, m]$ in $(M, \leq_L)$ and let m_1 and m_2 be any two elements of M. Since the atoms generate M both m_1 and m_2 can be written as products of atoms, each of which is a label used in P. Depending on the number of generators needed to write m_1 and m_2 draw a figure similar to the one shown with the factorization of m_1 into atoms written along the edges on the lower lefthand side and the factorization of m_2 into atoms written along the lower righthand side. The diagram shown assumes m_1 is a product of four atoms and m_2 is a product of three atoms. Next write m on each of the vertical arrows. Finally, we add labels to each of the other arrows, labeling one column at a time, working our way from the bottom to the top. If any two of the sides of a triangle are already labeled (which must be the bottom edge and the side edge) then there is a canonical way to add a label to the top edge, namely, find the unique $x \in P$ so that the label on the bottom edge matches the label on $(\hat{0}, x)$ and then label the top edge by the label on the interval $(x, \hat{1})$. This works because every label assigned to an interval of P also labels an interval of the form $(\hat{0}, x)$ for some x (Lemma 1.13). Once the diagram is completely labeled, let $n_1 \in M$ be the product of the labels on the upper lefthand edges and let $n_2 \in M$ be the product of the labels on the upper righthand edges. Because each triangle represents a relation in the monoid M, the diagram itself represents a proof that $m_1 \cdot n_1$ and $m_2 \cdot n_2$ represent the same element in M which is thus a right common multiple of m_1 and m_2. Finally, it is well-known result due to Ore [16] that a cancellative monoid in which every pair of elements has a right common multiple and a left common multiple embeds into some group and that the subgroup generated by its image is a well-defined object called its group of fractions. □

Left and right cancellativity is certainly necessary for a monoid to embed into some group and Ore's conditions (left and right cancellativity plus left or right common multiples) are known to be sufficient. A technical list of conditions on M that is both necessary and sufficient was established by Mal'cev [14], but the easy sufficient condition given above is all that is needed here.

Definition 1.16 (Garside group)**.** If M is a Garside monoid, then its group of fractions is called a *Garside group* and a group is *Garside* if it is the Garside group of some Garside monoid.

This definition of a Garside group does not really need all of the Garside axioms. In fact, Lemma 1.15 shows that even when M is merely a cancellative atomic monoid with a fundamental element, M has a well-defined group of fractions G and the natural map from M to G is an embedding. A glance back at the definitions reveals that a cancellative atomic monoid with a fundamental element is precisely an atomic monoid that satisfies axioms (G1), (G3) and (G4), but not necessarily (G2) or (G5). Lemma 1.15 is thus an example where the finiteness assumption and the lattice conditions are not really needed.

2. Combinatorial Garside structures

In this section we start with a labeled poset satisfying conditions only slightly more restrictive than those listed in Theorem 1.4 and show how to construct a Garside monoid containing it as its poset of divisors. Along the way we introduce the monoid, group and cell complex naturally associated with any labeled partially ordered set. Ok to here

Definition 2.1 (Combinatorial Garside structure)**.** A *combinatorial Garside structure* is a bounded, finite-height lattice P which is weakly-graded and has a balanced, group-like edge-labeling. The poset P is also presumed finite unless it is explicitly stipulated otherwise. If it is not known whether P is a lattice, but P satisfies all of the other conditions, then we say that P is *Garside-like*.

The two key differences are (1) weakly graded and (2) edge-labeling.

Definition 2.2 (Weakly graded)**.** Notice that a Garside monoid M is graded if and only if there is a monoid homomorphism $M \to \mathbb{N}$ which sends each atom to 1.

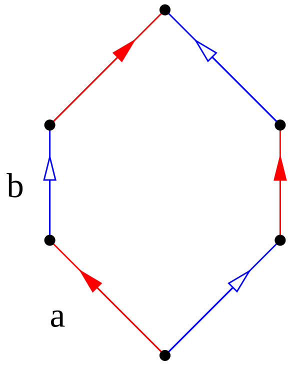

FIGURE 3. A simple edge-labeled poset in which the solid arrows are labeled a and the others are labeled b.

We call a Garside monoid M is *weakly graded* if there is a monoid homomorphism $M \to \mathbb{N}$ which sends each atom to a positive integer.

Definition 2.3 (Edge-labeled posets)**.** Let P be a bounded graded poset and let $I(P)$ be the set of intervals in P. Bounded and graded imply finite height but P can be arbitrarily "fat" (in Ziegler's terminology). In particular, P need not be finite. A *labeling of* P by a set S is a map $\lambda : I(P) \to S$

Combinatorialists typically only assign labels to covering relations, but edge labelings of this type can be easily converted into interval labelings as follows. First, label saturated chains with the words and then label intervals with languages. The set S in this case would be a collection of languages. For example, the "label" on $[\hat{0}, \hat{1}]$ for the righthand poset of Figure 3 would be $\{aba, bab\}$.

There is a close relationship between the Hasse diagram of $(M, \leq_L)$ and the right Cayley graph of M with respect to its atoms. Recall the definitions.

Definition 2.4 (Cayley graph)**.** Let M be a monoid with generating set A. The *right Cayley graph* of M with respect to A, denoted $\textsc{Cayley}(M, A)$, is defined as a directed graph whose vertices correspond to the elements of M and a directed edge is drawn from m to m' labeled by $a \in A$ if and only if $ma = m'$ in M.

Definition 2.5 (Hasse diagram)**.** If $(P, \leq)$ is a poset which locally has finite height, then the order relation is completely determined by its covering relations. These are recorded in a directed graph called its Hasse diagram. The *Hasse diagram* of $(P, \leq)$, denoted $\textsc{Hasse}(P)$, is a directed graph whose vertices correspond to the elements of P, and an edge is drawn from p to q if and only if $p < q$ is a covering relation in P.

Lemma 2.6 (Cayley graphs as Hasse diagrams)**.** *If M is a cancellative atomic monoid and A is its set of atoms, then the Hasse diagram of $(M, \leq_L)$ is the simple directed graph underlying the right Cayley graph* $\textsc{Cayley}(M, A)$.

Proof. Suppose that $m <_L m'$ and $ma = m'$. If this is not a covering relation, there must be some element m'' with $m <_L m'' <_L m'$. Converting these orderings into right multiplications, there must exist nontrivial elements $b, c \in M$ such that $m'' = mb$ and $m' = m''c = mbc$. Left canceling the m in $ma = mbc$ implies that a is not an atom. Since these steps can be reversed, this also shows that when a

is a not an atom, $m <_L m'$ is not a covering relation. Finally note that the atom involved is well defined (and hence the graph is simple). If $m <_L m'$ is a covering relation and a and b are atoms such that $ma = mb = m'$, then left cancellation shows that a and b denote the same atom in M. $\square$

By Lemma 2.6, the left divisibility ordering of a cancellative atomic monoid comes equipped with an implicit edge-labeling by its set of atoms.

Because of this correspondence, it makes sense to consider partially ordered sets in which each covering relation has been assigned a label from some set A.

Let A be a set, let $(P, \leq)$ be a poset and let $\textsc{Cover}(P)$ denote the set of ordered pairs (p, q) where $p < q$ is a covering relation in P. A *labeling* of P by A is an onto map $\lambda : \textsc{Cover}(P) \to A$. Let $\textsc{Chains}(P)$ denote the set of all saturated finite chains in P and let A^* denote the free monoid generated by the set A. Given a poset labeled by a set A it is natural to extend the labeling λ to a map $\lambda : \textsc{Chains}(P) \to A^*$ by assigning to each saturated finite chain the word obtained by concatenating the labels of its covering relations in the order they occur. Thus one element chains are sent to the empty word (i.e. the identity element in A^*) and a chain such as $C = (p_0 < p_1 < \cdots < p_k)$ is send to the word $\lambda(C) = \lambda(p_0, p_1).\lambda(p_1, p_2).\cdots.\lambda(p_{k-1}, p_k) \in A^*$.

Finally, a labeling λ is said to be *faithful* if it is injective on the set of edges of the form $(\hat{0}, u)$, thus determining a bijection $\mathcal{P} \setminus \{\hat{0}\} \leftrightarrow S$.

Definition 2.7 (Monoids and groups)**.** Given a labeled poset P, there is a monoid and a group naturally associated to P. Define $M(P)$ / $G(P)$ to be the monoid / group generated by the label set and relations equating the the labels on any two chains which start and end at the same elements. For example, the poset shown in Figure 3, the corresponding monoid and group presentations are $M = M(P) = \textsc{Mon}\langle a, b \mid aba = bab\rangle$ and $G = G(P) = \textsc{Grp}\langle a, b \mid aba = bab\rangle$.

Now view the labels as elements of the group G and define $K = K(P)$ as the quotient of the order complex of P where simplices with identical labels are identified respecting orientations.

Definition 2.8 (Balanced)**.** For any labeled poset P define $L(P) = \{\lambda(\hat{0}, p) : p \in P\}$, $C(P) = \{\lambda(p, q) : p, q \in P\}$, and $R(P) = \{\lambda(p, \hat{1}) : p \in P\}$. If the labels on the maximal chains are thought of as words read from $\hat{0}$ to $\hat{1}$, then $L(P)$ are the prefixes of these words, $R(P)$ are the suffixes and $C(P)$ are the subwords. More colloquially, these are subwords from the left right and center, hence the notation. The poset P is called *balanced* if $L(P) = R(P)$. Note that the notion of being balanced is quite different from that of having a symmetric labeling. For example, the labeled poset on the lefthand side of Figure 4 is *not* balanced because $ab \in L(P)$ but $ab \notin R(P)$. On the other hand, the poset on the righthand side is balanced.

When P is both balanced and group-like, then $L(P) = C(P) = R(P)$.

Definition 2.9 (Combinatorial Garside structures)**.** Let P be a bounded, weak-graded, edge-labeled poset. If P is balanced and group-like then it is called *Garside-like.* If P is also a lattice then P is a *combinatorial Garside structure.*

The weakly-graded condition is imposed here because a graded poset P leads naturally to a graded monoid M, but a poset P of finite height might lead to a monoid which is not locally of finite height.

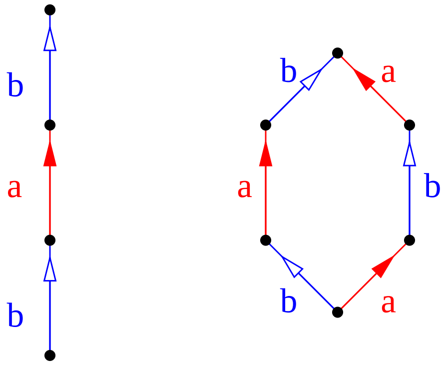

Figure 4. A non-balanced poset and a balanced poset

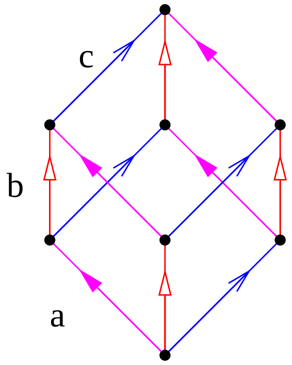

Figure 5. Finite Boolean lattices are Garside structures.

Finite Boolean lattices with the natural labeling are elementary examples of combinatorial Garside structures.

Example 2.10 (Boolean lattices)**.** As a first example, consider the Boolean lattice $\mathcal{B}_n$ viewed as all subsets of an n-element set under inclusion. If we label each covering relation by the element added, then it is clear that this label poset is balanced, group-like and a lattice. In this case, the monoid $M(P)$ is the free abelian monoid on n generators, the group $G(P)$ is the free abelian group on n generators, the order complex of P is a simplicial structure on the n-cube, and the complex $K(P)$ is a triangulation of the natural cell structure on the n-torus.

The following theorem establishes a bijective correspondance between graded Garside structures and bounded graded lattices which have faithful balanced labelings.

Theorem A (Equivalence)**.** *There is a natural bijection between combinatorial Garside structures and weakly graded Garside groups.*

Proof. By Theorem 1.4 and the remark after Definition 2.2, every Garside group contains a combinatorial Garside structure. It only remains to show that (1) every combinatorial Garside structure gives rise to a Garside group and that (2) these processes are inverses of each other.

(move a lot of this into lemmas or props)

For simplicity write $G = G(\mathcal{L}, \lambda)$, $M = M(\mathcal{L}, \lambda)$, and $\Delta = \Delta(\mathcal{L}, \lambda)$. We first observe that the grading function $\nu : S \to \mathbb{N}$ extends to grading homomorphisms $\nu_M : M \to \mathbb{N}$ such that $\nu_M^{-1}(0) = \{1\}$, and $\nu : G \to \mathbb{Z}$. In particular M is an atomic monoid, and (G1) is satisfied. Observe also that, by Lemma presentation, the presentation $\langle S \mid \mathcal{R} \rangle$ for the monoid M is equivalent to the following complemented presentation

$$\langle S \mid \lambda(\hat{0}, u).\lambda(u, u \vee v) = \lambda(\hat{0}, v).\lambda(v, u \vee v) \text{ for all pairs } u, v \in \mathcal{L} \rangle .$$

Here we understand $\lambda(u, u)$ to stand for the empty word! It is easily seen that this presentation satisfies the cube condition (on the left). It follows, by the usual argument (an induction founded on the grading of the monoid) that M is left cancellative and has least upper bounds with respect to the left-divsibility partial order. On the other hand, the same presentation may be written in the following form

$$\langle S \mid \lambda(u \wedge v, u).\lambda(u, \hat{1}) = \lambda(u \wedge v, v).\lambda(v, \hat{1}) \text{ for all pairs } u, v \in \mathcal{L} \rangle ,$$

which is complemented with respect to the right divisibility order, and satisfies the (right) cube condition. We observe by a completely analogous argument that M is right cancellative and has least upper bounds with respect to the right divisibility order. In particular, the monoid M satisfies the conditions of Öre (namely, left and right cancellativity and existence of upper bounds with respect to the left divisibility order) and therefore embeds in its group of fractions, which is canonically isomorphic to G. That is $\phi : M \to G$ is injective, and (identifying M with its image under ϕ) we have $G = M^{-1}M$. Moreover $M \cap M^{-1} = \{1\}$ (this also follows by extending the length homomorphism to a grading $h : G \to \mathbb{Z}$), and consequently M is the positive cone of a left-invariant partial order $\leq_L$ on G.

Since, by definition the order $\leq_L$ agrees with the left divisibility order on M, least upper bounds exist in $(G, \leq_L)$. For, if $x = a^{-1}b$ and $y = c^{-1}d$ are arbitrary elements of G, with $a, b, c, d \in M$, then, taking w to be an upper bound for a and c with respect to the right divisibility order on M, we have $wx, wy \in M$. Since least upper bounds exist in M, we may write $wx \vee wy$ for the least upper bound of wx and wy and observe that, by left invariance of $\leq_L$ on G, the element $x \vee y = w^{-1}(wx \vee wy)$ is the least upper bound of x and y.

Note that one may define a right-invariant partial order $\leq_R$ on G with positive cone M by setting $x \leq_R y$ if $yx^{-1} \in M$. This order agrees with the right divisibility order on M. It therefore follows, as above, that G has least upper bounds with respect to $\leq_R$. Since $x \leq_R y$ if and only if $y^{-1} \leq_L x^{-1}$ we deduce that $(G, \leq_L)$ has greatest lower bounds. Thus M is the positive cone of a left-invariant lattice order $(\leq_L)$ on G, and axiom (G2) is satisfied.

That M is atomic, (G1), follows from the grading on G, and axiom (G3) follows by the definition of M. Finally, (G4) follows from the fact that the labeling on $\mathcal{L}$ is balanced. For, the map $c : \mu \to \mu^{-1}\Delta$ defines a bijection of $[1, \Delta]$ such that c^2 is conjugation by Δ. It follows that Δ leaves $[1, \Delta]$ invariant, and since this subset generates M, leaves M invariant. □

3. Infinite Garside structures

In this section we show that the consequences of having a Garside structure extend to groups where the combinatorial Garside structure under consideration is infinite.

The next theorem answers the question: "What are Garside structures good for?"

Theorem B (Consequences). *If P is a combinatorial Garside structure and M, G and K are the monoid, group and complex derived from P then*

- $\textsc{Hasse}(P,S)$ *embeds into* $\textsc{Cayley}(M,S)$*,*
- $\textsc{Cayley}(M,S)$ *embeds into* $\textsc{Cayley}(G,S)$*,*
- *G is the group of fractions of M,*
- *the word problem for G is solvable, and*
- *the universal cover of K is contractible.*

This final property makes K an Eilenberg-MacLane space for G. It thus follows that the cohomology of G is equal to that of K, the cohomological dimension of G is bounded above by the dimension of K which in turn is equal to the height of P. It also follows that the group G is torsion-free.

As one can see from this theorem, the combinatorics of the labeled poset P to a large extent dominate and control the structure of the group G.

Proposition 3.1 (Cancellative Lattices). *If P is a possibly infinite combinatorial Garside structure and M is the monoid derived from P, then M is both left and right cancellative and $(M,\leq_L)$ and $(M,\leq_R)$ are lattices.*

Proof. (Quote [12] which does not need finiteness) (include the standard proof noting that it does not use finite) □

Proposition 3.2 (Meets and joins). *Let P be a possibly infinite combinatorial Garside structure and let M be the monoid derived from P. If the elements of P can recursively described, equality in P can be algorithmically tested and meets and joins of elements in P can be algorithmically produced, then meets and joins in the lattice $(M,\leq_L)$ and meets and joins in the lattice $(M,\leq_R)$ can be effectively computed.*

Proof. (standard inductive argument) □

Theorem C (Word problem). *Let P be a possibly infinite combinatorial Garside structure and let G be the group derived from P. If the elements of P can recursively described, equality in P can be algorithmically tested and meets and joins of elements in P can be algorithmically produced, then the word problem for G is solvable.*

Proof. (standard inductive argument) □

4. Completing partial orders

It is well-known that Dedekind used a method of cuts to complete the rationals to the reals. Less well-known is that H. M. MacNeille was able to generalize this technique of "Dedekind cuts" to show that every partially ordered set embeds in a complete lattice in an essentially unique and minimal way. The resulting complete lattice is called the Dedekind-MacNeille completion of the original partially ordered set. The material in this short section closely follows [9], particularly the second chapter on complete lattices.

Definition 4.1 (Dedekind-MacNeille completion)**.** Let $(P, \leq)$ be a partially ordered set and let A be a subset of P. The set A^u is defined as the set of elements $x \in P$ such that $a \leq x$ for all $a \in A$. Similarly, A^ℓ is the set of all $x \in P$ such that $x \leq a$ for all $a \in A$. In other words, A^u and A^ℓ are the collections of upper and lower bounds for A, respectively. The *Dedekind-MacNeille completion* of P is the collection of subsets A of P with $A = (A^u)^\ell$ ordered by set inclusion.

Theorem 4.2 (Properties of $\mathrm{DM}(P)$)**.** *For any poset P, its Dedekind-MacNeille completion $\mathrm{DM}(P)$ is a complete lattice. Moreover, there is an order-preserving embedding $\varphi : P \to \mathrm{DM}(P)$ of P into its Dedekind-MacNeille completion given by sending each element p of P to its principal ideal $\downarrow p$. In other words, $\varphi(p) = \downarrow p = \{q \in P | q \leq p\}$.*

Dedekind-MacNeille completions of partially ordered sets can be difficult to construct directly using the definition (particularly if P is infinite), but there is a characterization theorem which enables one to recognize when this minimal completion has been constructed by other means. The characterization is in terms of join-dense and meet-dense subsets.

Definition 4.3 (Join-dense and Meet-dense)**.** Let P be a subset of an ordered set Q. We say that P is *join-dense* in Q if every element of Q is the join of some subset of P. Similarly, P is *meet-dense* in Q if every element of Q is the meet of some subset of P.

The following characterization theorem is a restatement of Theorem 2.36 in [9].

Theorem 4.4 (Characterizing $\mathrm{DM}(P)$)**.** *Let P be an ordered set and let $\varphi : P \to \mathrm{DM}(P)$ be the order-embedding of P into its Dedekind-MacNeille completion defined above. The image of P under φ is join-dense and meet-dense in the complete lattice $\mathrm{DM}(P)$. Conversely, if L is a complete lattice and P is a subset of L which is both join-dense and meet-dense in L, then L is order-isomorphic to $\mathrm{DM}(P)$ via an order-isomorphism which agrees with φ on P.*

5. Working with nearly-lattice posets

As seen above, it is relatively easy to construct posets which are combinatorial Garside structures with the possible exception of the lattice condition. Moreover, there are nature situations (such as the noncrossing partition posets associated with the affine Coxeter groups) where the posets are not lattices but are close to being lattices. In this section we review how every poset can be completed to a lattice, a process called the Dedekind-MacNeille completion of a poset, and then discuss precisely how close this completion needs to be to the original in order for the consequences listed for Garside structures to still hold.

(talk about the extended labeling on $Q = DM(P)$, the free category C_Q and the complex K_Q)

Definition 5.1 (Nearly a lattice)**.** Let P be a possibly infinite Garside-like structure, let M_P and K_P be the monoid and complex derived from P, let $Q = DM(P)$ be Dedekind-MacNeille completion of P with the extended labeling, and let C_Q and K_Q be the category and complex derived from Q. We say that P is *nearly a lattice* when the natural map from M_P to C_Q is an injection and the natural inclusion of K_P into K_Q is a homotopy equivalence.

In practice, establishing the first condition typically involves showing that category diagrams with monoid boundaries can be promoted to monoid diagrams and establishing the second involves constructing a concrete deformation retraction from K_Q to K_P.

(There are the three main extensions we need to show: Props 5.2, 5.3 and 5.5. The rest of this section is then easy corollaries.)

Proposition 5.2 (Cancellative)**.** *If P is a possibly infinite Garside-like structure, $Q = DM(P)$ is the Dedekind-MacNeille completion of P with the extended labeling and C_Q is the free category derived from Q, then C_Q is both left and right cancellative.*

Proof. (mimic the earlier proof) □

Proposition 5.3 (Contractible)**.** *If P is a possibly infinite Garside-like structure, $Q = DM(P)$ is the Dedekind-MacNeille completion of P with the extended labeling and K_Q is the cell complex derived from Q, then the universal cover of K_Q is contractible.*

Proof. (mimic the earlier proof) □

Theorem 5.4 (Consequences)**.** *Let P be a possibly infinite Garside-like structure that is nearly a lattice. If M, G and K are the monoid, group and complex derived from P then*

- HASSE(P, S) *embeds into* CAYLEY(M, S),
- CAYLEY(M, S) *embeds into* CAYLEY(G, S),
- *G is the group of fractions of M, and*
- *the universal cover of K is contractible.*

This final property makes K an Eilenberg-MacLane space for G. It thus follows that the cohomology of G is equal to that of K, the cohomological dimension of G is bounded above by the dimension of K which in turn is equal to the height of P. It also follows that the group G is torsion-free.

Proof. (easy cor) □

Finally, we extend the proof of Theorem C to cover the solvability of the word problem to the category derived from a Dedekind-MacNeille completion.

Proposition 5.5 (Word problem)**.** *Let P be a possibly infinite Garside-like structure, let $Q = DM(P)$ be its Dedekind-MacNeille completion with the extended labeling and let C_Q be the free category derived from Q. If the elements of Q can recursively described, equality in Q can be algorithmically tested and meets and joins of elements in Q can be algorithmically produced, then the word problem for C_Q is solvable.*

Proof. (mimic the earlier proof) □

As a final corollary, conditions sufficient to

the following hypotheses are sufficient to solve the word problem in the group derived from a possibly infinite Garside-like structure.

Theorem 5.6 (Word Problem)**.** *If P be a possibly infinite Garside-like structure which is nearly a lattice, let $Q = DM(P)$ be its Dedekind-MacNeille completion and let G be the group and derived from P. If the elements of Q can recursively described, equality in Q can be algorithmically tested and meets and joins of elements in Q can be algorithmically produced, then the word problem for G is solvable.*

Proof. Let M be the monoid derived from P and let C be the free category derived from Q. Since G is the group of fractions of M, to solve the word problem for G, it is sufficient to solve the word problem for M. By the nearly lattice assumption, M is included as a submonoid of C so it is sufficient to solve the word problem for C since the word problem for M is simply a special case. Finally, the word problem for C is solvable by Proposition 5.5. $\square$

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Department of Mathematics, University of Oklahoma, Norman, OK
E-mail address: `nbrady@math.ou.edu`

Institut de Mathématiques, Université de Bourgogne, UMR 5584 du CNRS – BP 47 870, 21078 Dijon, France
E-mail address: `jcrisp@u-bourgogne.fr`

Dept. of Math., California Polytechnical Institute, San Luis Obispo, CA
E-mail address: `akaul@calpoly.edu`

Department of Mathematics, University of California, Santa Barbara, CA 93106
E-mail address: `jon.mccammond@math.ucsb.edu`