GARSIDE STRUCTURES FOR ARTIN GROUPS

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Abstract. In this article we define a partially ordered set for each finitely generated Artin group A. In many case – and conjecturally in all cases – this poset provides A with a Garside structure. These structures are similar to the dual Garside structures recently found in the finite-type cases. In particular, they provide a new presentation for the Artin groups, they solve the word problem, and they lead to a finite dimensional Eilenberg-Maclane space.

Contents

Despite their derivation from Coxeter groups and the intensive amount of work that the braid groups have inspired, Emil Artin's generalized braid groups have remained frustratingly mysterious. In this article we propose that researchers focus their attention on a particular collection of partially ordered sets which, conjecturally at least, will provide weak Garside structures for all finitely generated Artin groups. The establishment of our three conjectures would imply - for the first time - that an arbitrary Artin group has a decidable word problem, has finite cohomological dimension, has a finite Eilenberg-Maclane space, is a (weak) Garside group, is torsion-free, is linear, and is the group of fractions for a natural monoid presentation. In contrast with their consequences, the conjectures themselves seem rather mild. Specifically, we conjecture that the Tits section and the Paris representation extend to slightly larger generating sets and that certain partially ordered sets have least upper bounds.

From talking with John Crisp we now have an algebraic representation and a topological representation which are cleanly defined. The topological and algebraic representations are easily seen to be equivalent and the algebraic one is easily seen to define the correct group.

On the train to London, we worked on the lattice condition and our proof was a bit bogus because we only paid attention to the increasing portion of the poset when we really needed to focus on the shrinking bit. The big bit is easy, the small bit might take some work. In fact, we might need to switch back to plan A (one speculative article by me followed by other articles rather than a big article by the four of us). We'll see what happens in the next few weeks. I still like the algebraic/topological approach. The elements are double cosets. The subgroup on one side is a product braid groups and the subgroup on the other is a subgroup generated by powers of the Birman-Ko-Lee generators.

The structure of the article: After establishing the basic definitions in Section 1, and describing the three main conjectures in Section 2, we present the major constructions in Section 3. Section 4 is devoted to the key example of the free groups, viewed as an Artin group over a discrete graph. Section 6 extends this discussion to the general situation.

In the later sections we discuss ideas, remarks and implications regarding the Tits section (Section 7), the Garside structure (Section 8), the resolution of the finite Eilenberg-Maclane problem (Section 9), and the Paris representation (Section 10). Finally, Section 11 contains a few concluding remarks.

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More specifically, the idea of focusing on factorizations of the Coxeter element in the corresponding Coxeter group and the illustration of this process using the free group of rank 2 emerged from conversations with Anton Kaul during the Cannonfest in Park City, Utah (early June 2003). The necessity of restricting to the image of chains under the braid group action, the partial results on 3-generator Artin groups, and the idea behind the deformation retraction to a finite Eilenberg-Maclane space are from conversations with Noel Brady in Seville, Spain (mid-June 2003). In Braga, Portugal (late June 2003) Ruth Corran generously shared her expertise on centralizers in Artin monoids as we investigated the possibilities of a Garside lift (Conjecture A). Finally, the computer algebra package GAP[citeGap] has been a constant traveling companion, tirelessly checking examples and producing additional revealing patterns for further investigation. Lastly, as I was writing this article, I was informed that Franscoise Digne has recently considered some similar constructions for certain Artin groups. I am unsure at this point whether the structures he considers are identical with the ones described below, but, given the naturality of their definition, I would not be at all surprised if this indeed turned out to be the case.

1. Basic definitions

In this section we establish our notation. All of these are fairly standard and have been included for completeness.

Definition 1.1 (Ordered labeled graphs)**.** Let Γ be a finite graph without loops or multiple edges. A *labelling* on Γ will be an assignment of integers to its edges with each label bigger than 1. For those readers more familiar to Coxeter notation, notice that edges labeled 2 are drawn, but that edges labeled ∞ are not. An *ordering* of Γ is a linear ordering of its vertices.

Definition 1.2 (Artin relations)**.** Following a suggestion by Walter Neumann, we will use the notation $(a,b)^n$ to denote the first n letters of $(ab)^n$. Thus $(a,b)^2 = ab$, $(a,b)^3 = aba$ and $(b,a)^3 = bab$.

Definition 1.3 (Artin groups)**.** For each finite labeled graph Γ, there is an Artin group A_Γ whose standard presentation has a generator for each vertex and a relation $(a,b)^n = (b,a)^n$ whenever there is an edge labeled n connecting vertices a and b.

Definition 1.4 (Coxeter groups)**.** For each finite labeled graph Γ, there is a Coxeter group W_Γ which has the same presentation as the Artin group A_Γ, but with an additional relation $a^2 = 1$ for each vertex $a \in \Gamma$.

Definition 1.5 (Coxeter element)**.** For each ordered labeled graph Γ, there is a Coxeter element δ obtained by multiplying the standard generators in the order prescribed by the ordering of Γ.

Figure 1. An ordered labeled graph

Example 1.6. If Γ is the ordered labeled graph shown in Figure 1, then the Artin group A_Γ has a standard presentation XXX and the Coxeter group W_Γ has a standard presentation YYY.

2. Main conjectures

For each finite ordered labeled graph Γ (used to define an Artin grup A_Γ and a Coxeter group W_Γ) we will define a pair of bounded, graded, self-dual, edge-labeled posets of finite height, P_Γ^A and P_Γ^W, a finite-dimensional simplicial complex K_Γ, and a monoid M_Γ. All of these objects will be explicitly defined in the next section. For each such Γ we have the following three conjectures.

Conjecture A (Tits section)**.** *The posets P_Γ^A and P_Γ^W are naturally isomorphic.*

Conjecture B (Garside structure)**.** *The poset P_Γ^A is a lattice.*

Conjecture C (Paris representation)**.** *The Paris representation of A_Γ is injective on the positive monoid M_Γ.*

The poset P_Γ^A will produce a complex K_Γ whose fundamental group is the Artin group A_Γ but since A_Γ is used in its definition - and the word problem for an arbitrary Artin group is not known to be solvable, P_Γ^A will be difficult to work with directly. The poset P_Γ^W, on the other hand, is derived from the corresponding Coxeter group about which quite a bit is known. For example, its word problem is decidable and it acts discretely cocompactly by isometries on a piecewise Euclidean CAT(0) complex, hereby providing the researcher with a plethora of tools. Conjecture A is thus crucial to our ability to work concretely with these posets. The reasons why this conjecture appears more tractable than tackling the word problem directly will be given in Section 7.

Conjecture B would solve the $K(G,1)$ problem for Artin groups and Conjecture C would prove that all Artin groups are linear, but without Conjecture A, it will be difficult to establish either of these conjectures.

Remark 2.4 (Previous results)**.** The only previously known results of which we are aware that apply to arbitrary finitely generated Artin groups are [citeCrispParis] and [citeParis]. These show that the subgroup generated by squares and the positive monoid, respectively, behave as expected. In both cases, the proof proceeds by a sequence of representations (into mapping class groups of surfaces and infinite dimensional matrix groups, respectively).

Remark 2.5. Some remarks on the conjectures. To prove the Tits section, the key steps seem to be
1a) that no squares show up in the braid action on F_n.
1b) no two letter subword has more one change of sign.
1c) these properties are preserved as we move towards an a-parallel geodesic.
2. Lattice (hardest, maybe needs CAT(0))
3. Positive monoid embeds. (easiest?)

Tools: The four hammers avaiable for our use are
- injection of Braid_n into $\text{Aut}(F_n)$ (Krstic)
- CAT(0) structure on the Coxeter complex (Moussong)
- Tits conjecture for squares (Crisp-Paris)
- Artin monoid injects (Paris)

The relevance of the first result will become appearant once the posets have been defined. Sava Krstic showed that the braid group Braid_n embeds in $\text{Aut}(F_n)$ by conjugating in a (fairly) obvious way. Somehow this should be the key bit of info. I might even be able to use the delta element to squeeze out everything from just this. This has to work because there's so much to use.

3. Algebraic constructions

In this section we define our main algebraic construction. We begin by descibing the action of the braid groups on ordered generating sets.

Definition 3.1 (Bases)**.** Let F_n be a free group and let $\langle x_1, x_2, x_3, \ldots, x_n\rangle$ be a basis for F_n. This basis is actually an *ordered basis* where the ordering is given by the subscripts. The *ordered product* of this ordered basis is $\delta = x_1 x_2 \cdots x_n$. If we view F_n as an Artin group then δ is merely the Coxeter element for the ordered labeled graph with n vertices and no edges. The collections of automorphisms of F_n form a group $\text{Aut}(F_n)$. The subgroup we will be most interested in will be the group of symmetric automorphims. An automorphism is *symmetric* if each standard basis element x_i is sent to a conjugate of some x_j. It is *pure symmetric* if each x_i is sent to a conjugate of itself. The symmetric automorphisms and pure symmetric automorphisms form subgroups of $\text{Aut}(F_n)$ which we will denote Σ_n and $P\Sigma_n$.

Definition 3.2 (Braid action)**.** Consider the collection $\mathcal{T}$ of ordered bases of the free group F_n. For example, if $\langle x_1, x_2, x_3, \ldots, x_n\rangle$ is one ordered basis, then $\langle x_1x_2, x_2, x_3, \ldots, x_n\rangle$ is another.

Recall that the braid group $\textsc{Braid}_n$ is the Artin group defined by a ordered labeled complete graph on $n-1$ vertices with a 3 labeling successive vertices and a 2 on every other edge. Let σ_i denote the i-th generator of in its standard presentation. The braid group acts on this set as follows. (add this in)

It only remains to check that σ_i and σ_j commute when $|i-j|>1$ and that they satisfy a braid relation ($aba = bab$) when $i+1=j$. Both identities are routine to verify. Notice also that the action by the braid group preserves the ordered product of the basis elements. The deeper fact, first shown by Magnus is that ordered bases obtaining by braiding the standard ordered basis are the only symmetric bases whose ordered product is the Coxeter element $\delta = x_1 x_2 \cdots x_n$. check reference

Definition 3.3 (Free Garside structure)**.** (define the poset P_n)

Definition 3.4 (Free Garside complex)**.** (define the complex K_n)

Definition 3.5 (Free Garside monoid)**.** (define the monoid M_n)

The main goal of this section is to define the braid group action on ordered generating sets and to record its main properties.

Definition 3.6 (Movable type)**.** Let P be a poset with edge labels. It has *movable type* if given a finite chain, from x to y and any z above y (or below x) there is a chain with the same edge labels ending at (starting at) z.

Intervals in the Cayley graph of groups generated by twist closed generating sets will have this property. This is the same as right divisors = left divisors (locally).

The positive word trick: Given a word in F_n, we can rewrite it as a power of δ times a positive word in the conjugates of the standard generators by δ. Call this a_i, b_i etc. where a_0, b_0 etc are the standard ones. All the deltas can then be moved to the front at the cost of changing the subscripts. Thus we get a power of δ times a positive word in the a_i, b_i, ...

This trick applies equally well to the braid groups.

Idea: Maybe even in the finite-type cases, this idea of restricting to twist closed sets needs to be tried. There's an outside possibility that in the D_4 and F_4 cases, these generating sets are strictly smaller than the ones obtained by performing *all* factorizations.

Let M be a monoid with an ordered generating set $\{x_1, x_2, \ldots, x_n\}$ and a presentation $\langle X | U_i = V_i \rangle$ where $|U_i| = |V_i|$ for all i. (weaken this so that Coxeter groups are included)

– I think what I want here is a group with a length function that is unchanged by conjugation. For example, consider the translation length of an element on the *vertices* of the Cayley graph. This is clearly unchanged by conjugation and it is 1 on generators.

– what's the easiest way to describe these posets and their images into Artin groups and Coxeter groups? Does Sava Krstic's proof of injectivity extend (in principle) to arbitrary Artin groups? We need

– Notice that B_n acts on the CAT(0) space K_W^n. In the case of a free group, we have K_W is an undirected n branching tree, and B_n acts on the direct product of n copies. Goal one is to show that this action is faithful (hopefully by showing that there are no big syllables.

Let G the corresponding group. There is a natural class of additional generating set that are obtained by letting the braid groups act on this ordered set. The resulting orbit will be called *twist closed*. These can also be collected into a graded poset.

Lemma 3.7. *The braid group* BRAID_n *acts on any ordered generating set and thereby creates a graded self-dual poset which is twist closed and generates the group. If this poset is a lattice then* G *is a (weak) Garside group.*

4. Free groups

The main goal of this section is to describe the free Garside structure on a finite generated free group.

Let $\langle x_1, x_2, \ldots, x_n\rangle$ be an ordered basis for the free group F_n. Since the braid groups act freely on ordered bases of F_n by conjugation (Krstic), there is a well-defined poset P_n whose maximal chains are labeled by these braid-many ordered bases.

The ends of the bases have the following nice properties:

1. either length 1 or palindrome

2. if "a" and "aA" cooccur then "a" is to the right. (similarly for the other letters and for "Aa")

3. "aA" and "Aa" never cooccur. 4. the conjugating letters occur in increasing order.

Let $P = \langle a_1, a_2 | a_1^2 = a_2^2 = 1\rangle$. This is the infinite dihedral Coxeter group W acting on the real line. There are infinitely many reflections a_i, $i \in \mathbb{Z}$.

The new presentation is $Q = \langle x, a_i | a_i a_{i+1} = x\rangle$. This is a conjugacy closed generating set of W, so the interval in the Cayley graph has the movable type property.

The universal cover is $T_\infty \times \mathbb{R}$ with a very strange F_2 action.

A presentation of the topological definition and approach to partially order fundamental sets and presentations for arbitrary Artin groups.

Let D^2 denote the unit disk in $\mathbb{R}^2$ and let $\{p_1, p_2, \ldots, p_n\}$ denote a set of n distinct points in D^2, as shown in the figure. We write $D_\star = D^2 \setminus \{p_1, p_2, \ldots, p_n\}$. Let $\mathbf{a} = (0, 1)$ and $\mathbf{b} = (0, -1)$ denote the top and bottom points of the boundary $\partial D = \partial D^2$. We shall take $\mathbf{b}$ as a basepoint for the fundamental group $\pi_1(D_\star)$. We choose n intervals $A_1, .., A_n$ which are smoothly embedded in D^2 and disjoint except at $\mathbf{a}$ and such that A_i joins the point $\mathbf{a}$ to p_i. We identify the group $\pi_1(D_\star)$ with the free group $F_n = F(x_1, x_2, .., x_n)$ where x_i is represented by any loop at $\mathbf{b}$ which crosses A_i exactly once travelling in a clockwise direction around p_i and is disjoint from all other α_j.

Definition 4.1 (Cut curves)**.** By a *cut-curve* or *curve* on $D_\star$ we shall mean a smoothly embedded interval c in $D_\star$ which meets ∂D precisely at its endpoints, and which separates the boundary points $\mathbf{a}$ and $\mathbf{b}$.

We shall say that two cut-curves are isotopic if they are (ambient) isotopic relative to $\{p_1, .., p_n, \mathbf{a}, \mathbf{b}\}$ (i.e: isotopic in $D_\star$ relative to $\{\mathbf{a}, \mathbf{b}\}$). We denote by $[c]$ the isotopy class of a curve c, and write $\mathcal{C}$ for the set of all isotopy classes of cut-curves in $D_\star$.

Observe that any cut-curve c separates $D_\star$ into two regions, an upper region containing $\mathbf{a}$ and a lower region containing $\mathbf{b}$, and induces, in particular, a partition of the points $\{p_1, .., p_n\}$ into two sets. In general we shall say that the contents of the region containing $\mathbf{a}$ lie *above* c and the contents of the region containing $\mathbf{b}$ lie *below c* . For each curve c, we write $\deg(c)$ for the number of points p_i which lie below c. Clearly this number is invariant under isotopy, and so defines a degree function on $\mathcal{C}$ by $\deg([c]) = \deg(c)$.

Let c_1, c_2 be two curves. We say that c_1 and c_2 are in *minimal position* with respect to one another if they do not cobound any disk regions ("bigons") in $D_\star$. (This includes triangular regions against the boundary). Any such disk regions can always be removed by modifying just one of the two curves in its isotopy class without changing the other. That is to say that, for any two curves c_1, c_2 we can always find c_1' such that $[c_1'] = [c_1]$ and c_1' is in minimal position with respect to c_2.

Definition 4.2 (Arc order)**.** Let c_1, c_2 be two curves. We say that $[c_1] \leq [c_2]$ if c_1 is isotopic to a curve which lies below c_2. It is easily checked that this defines a partial order on the set $\mathcal{C}$ of cut-curve classes.

Note that if c_1 and c_2 are in minimal position with respect to one another then $[c_1] \leq [c_2]$ if and only if c_1 lies below c_2 (in particular they are disjoint). Note also that the function $\deg : \mathcal{C} \to \{1, .., n\}$ is a strict order preserving map, a grading on the poset $(\mathcal{C}, \leq)$.

Proposition 4.3. *The graded poset* $(\mathcal{C}, \leq)$ *is a lattice.*

Proof. Let $x = [c_1]$, $y = [c_2]$ be arbitrary elements of $\mathcal{C}$. Suppose that curves c_1 and c_2 are in minimal position with respect to one another. Now consider how the union of c_1 and c_2 cut $D_\star$ up into connected regions. There is a unique lowermost such region R which lies below both c_1 and c_2 (and contains the point $\mathbf{b}$), and another uppermost region R' which lies above both curves (and contains $\mathbf{a}$). Let c, resp. c', denote the curves which skirt along the part of the boundary of R, resp. R', lying in the interior of $D_\star$ (i.e. NOT along ∂D). Then we claim that $x \wedge y = [c]$ and $x \vee y = [c']$.

We check just the first of these two claims. Suppose that c_0 represents a common lower bound for x and y. Then, by a sequence of bigon removing isotopies, we may choose the representative c_0 to be in minimal position with respect to both c_1 and c_2. But now, since it represents a common lower bound, c_0 lies below both c_1 and c_2, and hence lies below the curve c. □

The elements of $\mathcal{C}$ correspond to certain special elements of F_n, as follows. Given a cut-curve c let γ be the loop at $\mathbf{b}$ which traverses the boundary of region of $D_\star$ which lies below c in a clockwise direction. Then γ represents an element of F_n for which a word in the generators x_i may be read by running along c from left to right. We may write $[\gamma] = [c]$ without confusion. (Easy exercise check that if γ, γ' are homotopic then the cuts c, c' were isotopic). Thus the minimal element of $\mathcal{C}$ corresponds to the identity in F_n, and the maximal element to the group element $x_1 x_2 .. x_n \in F_n$.

Any maximal chain in $\mathcal{C}$ can be given by a sequence of cut-curves $c_0, c_1, c_2, .., c_n$ where $\deg(c_i) = i$, for each $i = 0, 1, .., n$, and where each c_i lies below c_{i+1}. Such a sequence can be associated to an "arc system".

Definition 4.4. An *arc system* is a family of n intervals $\alpha = \{\alpha_1, .., \alpha_n\}$ smoothly embedded in D^2 such that

: (i) α_i joins $\mathbf{b}$ to $p_{\sigma(i)}$, where σ denotes a permutation of $\{1, .., n\}$;
: (ii) $\alpha_i \cap (\{p_1, .., p_n\} \cup \partial D) = \{\mathbf{b}, p_{\sigma(i)}\}$;

: (iii) the α_i are mutually disjoint except at the point $\mathbf{b}$. Two arc systems are called *isotopic* if they are ambiently isotopic relative to $\{p_1, .., p_n, \mathbf{b}\}$;

: (iv) the α_i are labelled in in clockwise order around the basepoint $\mathbf{b}$.

Given a sequence of cut-curves as above one may construct an arc system (which is unique up to isotopy) such that α_1 is below c_1, α_2 is below c_2, etc..

Lets write $w(\alpha_i)$ for the element of F_n represented by $\alpha_i x_{\sigma(i)} \alpha_I^{-1}$ (more precisely, the loop in $D_\star$ based at $\mathbf{b}$ which goes out along α_i clockwise around the endpoint, then return back along α_i). Then, if α is the isotopy class of arc systems corresponding to the chain $c_0, c_1, .., c_n$ as above, we have (in F_n) that $w(\alpha_i) = [c_{i-1}]^{-1}[c_i] = w_i$, for each $i = 1, .., n$. The elements $w_1, w_2, .., w_n$ form an ordered free basis for F_n. Moreover, any such basis is conjugate to the standard free basis $x_1, .., x_n$ (taking w_i to x_i) by a diffeomorphism of $D_\star$.

In fact we have bijective correspondances

$$
\begin{aligned}
\{\text{maximal chains in } \mathcal{C}\} &\leftrightarrow \{\text{isotopy classes of embedded arc systems}\} \\
&\leftrightarrow \{\text{"geometric" ordered free bases}\} \\
&\leftrightarrow \{\text{elements of } MCG(D_\star, \partial D) \cong B_n\}
\end{aligned}
$$

We now define a partially ordered set for an arbitrary Artin group, given by defining graph Γ with totally order vertex set $x_1, .., x_n$, and relator indices m_{ij}.

For each $1 \leq i \neq j \leq n$ for which $m_{ij} \neq \infty$ choose a smoothly embedded arc β_{ij} joining p_i to p_j whose interior is disjoint from all of the intervals $A_1, .., A_n$ and disjoint from ∂D. Define the half Dehn twist τ_{ij} along β_{ij} and write T_{ij} for the element of $B_n \cong MCG(D_\star)$ represented by $\tau^{m_{ij}}$. Finally write $H_\Gamma =< T_{ij} : m_{ij} \neq \infty >$ for the subgroup of B_n generated by these twists.

Definition 4.5. For Γ as above, we define $\mathcal{P}_\Gamma^{\text{top}} = \mathcal{C}/H_\Gamma$. For two elements $x, y \in \mathcal{P}_\Gamma^{\text{top}}$ we write $x \leq y$ if there exist elements X, Y of $\mathcal{C}$ representing x, y respectively such that $X \leq Y$.

The big question is whether $\mathcal{P}_\Gamma^{\text{top}}$ is really a lattice (???)

5. Finite-type Artin groups

In this section we prove that the entire program works for all finite-type Artin groups, and in particular, that this method recovers their dual monoid presentations.

(Actually, all of the results in this section are proved in Bessis, roughly along these lines)

Recall that the usual definition of the poset P_Γ^{alg} in the finite-type case is as all minimal factorizations of the coxeter element in terms of all reflections. The twist closed factorizations is a subposet of this one. Thus, we only need to show that there does not exist a twist closed subposet.

Let Γ be the labeled graph for a finite irreducible Coxeter group. If the edges labeled 2 are temporarily ignored, then the result is a tree, and thus bipartite. If the vertices are 2-colored, say red and blue, then we will order the vertices so that all of the vertices of one color are listed before those of the other. Call the product of the all the blue reflections τ_1 and the product of all of the red reflections τ_2. Notice that the order in which we take the product is irrelevant since the blue (red) reflections pairwise commute.

Lemma 5.1 (Reflections)**.** *If* Γ *is the labeled graph for an irreducible finite Coxeter group with a bipartite ordering, then every reflection* r *in* W_Γ *is equal to* $s^{(\delta^k)}$ *for some standard reflection* s *conjugated by some positive power* k *of the Coxeter element* δ*.*

Proof. The proof of this is case by case. The low cases (say, of rank at most 9) can be easily checked by GAP. Since the only irreducible graphs beyond that are of types A_n, B_n and D_n, we can use their representations as noncrossing partitions to check that this is true (which it is).

(I've checked this explicitly for type A_n and B_n. For D_n it is true for $n \leq 9$ and I haven't bothered to check the noncrossing partition interpretation in general) □

(remark: this is proved in Bessis Lemma 2.2.2)

Lemma 5.2. *If* Γ *is the ordered labeled graph for a finite Coxeter group, then every reflection* r *in* W_Γ *occurs as a label in* P_Γ^{alg}*.*

Proof. Suppose Γ is irreducible. Since all of the Coxeter element are conjugate, the ordering on Γ can be assumed to be the bipartite ordering. If we pull the top element to the bottom, the result is to conjugate it by δ. Iterating this produces every reflection by Lemma 5.1.

When Γ is reducible, we can pick an ordering so that the irreducible components occur consecutively. If the reflection we wish to produce and it belongs to the i-th irreducible component, then we simply twist that portion of the ordered basis in order to produce it. □

Actually, more is true. For every reflection r, the interval above the edge labeled r in P_Γ^{alg} can be factored using the reflections which bound a particular chamber in a "sub" Coxeter complex. (i.e. the one obtained when we quotient out the one direction fixed by $r\delta$.)

Theorem 5.3 (Finite case)**.** *If* Γ *is the ordered labeled graph for a finite Coxeter group, then the edge-labeled poset* P_Γ^{alg} *is label isomorphic to the lattice* NC_Γ*, and hence isomorphic to* P_Γ^{top} *as well.*

Proof. The proof is by induction. At this point, we use the strong claim to see that every reflection bounds a chamber s.t. its reflections in the right order gives the coxeter element. This should allow us to view the interval above this reflection as a twist poset for a smaller rank coxeter group, completing an induction.

Given the fact that every reflection shows up in the twist subposet, I'd be flabbergasted if there was a proper subposet which was twist closed.

The final conclusion follows from the fact that NC_Γ defines A_Γ [Bessis,Brady-Watt] and hence "lifts" and embeds. □

Oh.. and one last thought about the double coset representation. Both B_n and H have a very special property. The map sending each standard generator to its own inverse extends to an automorphism. This is because the standard relators are invariant under a uniform reversal of arrows. I think this should allow us to move the H to the other side somehow and talk about cosets of $(B_k x B_l)/H$ somehow.

p.p.s. the nonempty intersection you mentioned is just the common refinement $(B_k \times B_l \times B_m)gH$ I was talking about using to describe chains in $\mathcal{P}_\Gamma^{\text{top}}$.

6. Non-crossing partitions

By analogy with the type A_n case, we will call the resulting posets the poset of non-crossing partitions. Actually in Section 8 we will show that all of these posets are lattices.

The main result of this section should be that there is a Garside lift from P_W to P_A. First we need to define these objects.

Let A be an Artin group defined by the labeled graph Γ, and let W be the correspondiing Coxeter group. Let S be the standard generators of W with some ordering given.

, let T be the set of all reflections in W, and consider the Cayley graph of W with respect to T. Define δ to be the product of the elements of S in some order. δ is called the Coxeter element and it's important to fix this once and for all.

Definition 6.1 (Braid twisting)**.** Let $\phi : B_{n-1} \to \text{Aut}(F_n)$ be the standard representation of the braid group as the set of all factorizations of the element $x_1x_2\cdots x_n$ into conjugates of the standard basis elements (Krstic). This defines a poset as follows.

Definition 6.2 (δ-poset)**.** Let P be a portion of the smallest subgraph of the Cayley graph $\text{Cayley}(W,T)$ which contains all geodesics from 1 to δ. We can turn this labeled graph into a poset by orienting all edges away from 1. Every interval $[x, y]$ is labeled by the element $x^{-1}y$ of W. Which portion of this poset is selected is based on the braid twisting defined below.

Definition 6.3 (Artin complex)**.** Let K_0 be the geometric realization of P, and notice that the edge labeling of P produces a labeling of the oriented 1-skeleton of K_0. This will be our fundamental domain of our complex. Quotient this complex by the following identifications. Every pair of simplices in K_0 with identically labeled 1-skeletons will be identified.

Lemma 6.4. *The edge leaving $\hat{0}$ in P are labeled bijectively by the elements in T. (false. I'm now restricting T to whatever occurs)*

Lemma 6.5 (Self-duality)**.** *The poset P_Γ has an order reversing automorphism and hence is self-dual.*

Proof. Send every element above $\hat{0}$ to its "complement" below $\hat{1}$. (This is the same proof based on movable type as in the finite case.) □

7. Garside lifts and the fundamental group

The main goal of this section is to explicate the consequences of Conjecture A. In particular, to show that the existence of a Garside lift of the expanded positive monoid implies that the fundamental group of K_Γ is A_Γ.

Theorem 7.1. *The image of P_Γ in W_Γ is isomorphic to its image in A_Γ for every finite ordered labeled graph Γ.*

The key lemmas along the way are that

Lemma 7.2. *Every reduced word and every reduced partial product in a braid twist of the standard basis of the free group is "square-free" in the sense that every pair of adjacent letters represent distinct generators.*

Proof. The proof will proceed by induction of the length of the braid twisting. To start the induction notice that the assertion is clearly true for the standard basis $(a_1, a_2, \dots, a_n)$. Suppose the lemma is true for all braid twisting of length at most k and consider the induction that $\pi(a_1^{u_1}, a_2,^{u_2}, \dots, a_n^{u_n})$ where each word is reduced in the free group and π is some permutation of p$[n]$. The result of twisting this basis by σ_i is a new basis where $n-1$ of the basis elements haven't changed, and the new element is $uaUvbVuAU$. If Uv has a reduction the Vu also has a reduction and this is a conjugate of a small conjugate. Thus we can assume that Uv doesn't have a cancellation.

(use induction on braid length) □

Corollary 7.3. *The Garside lift is well defined for free groups.*

The previous lemma is the first step of an induction on the length of geodesic reductions in an arbitrary

Lemma 7.4. *(geodesic reductions and no wiggly bits)*

• We still need to show that there is a Garside lift (similar to the Tits lift) for these posets (and positive monoids). (Corran, Bridson, GAP)

Theorem 7.5 (Garside lifts exist)**.** *For every defining graph* Γ*, the Garside lift* $\gamma : P_W \to P_A$ *is well-defined.*

Theorem 7.6. *If Conjecgture A holds for* Γ*, then* $\pi_1(K_\Gamma)$ *is isomorphic to the Artin group* A_Γ*.*

Proof. This proof should be a quick consequence of the movable type property. Here's an outline:

1 Using movable type we can show that every relation in the standard presentation is trivial in $\pi_1 K$. Thus we have a map $A \to \pi_1 K$.

2 The reverse map exists so long as every edge in the poset is in the image of the standard product under movable type moves. This gives a map $\pi_1 K \to A$.

The composition of these two maps $A \to A$ is the identity on the generators, therefore an isomorphism on A. Moreover, **2** implies that the first map is onto, so $\pi_1 K$ is isomorphic to A. □

Corollary 7.7. *Every finite generated Artin group has a decidable word problem.*

Proof. (The proof should basically proceed by showing how to create a (algorithmically produced) finite normal form. The trick is that producing all normal forms is an infinite process (infinite number of generators, infinite number of relations, infinite set of "reduced" words). But with care, we should be fine.) □

Alternatively, we could wait until after we show linearity. Then, using L. Paris' approach try to show that the extended positive monoid embeds. Then use the fact that have lcm to get every element in the form uv^{-1}. Actually, all we'll need is something much much weaker. All we'll need is that $Mn \cap Mm \neq \emptyset$ (via Stuart's talk).

What about the conjugacy problem? Can we do it as well?

8. Garside structures

The main result of this section should be that P_W is a lattice, and hence a Garside structure. As a nearly immediate Corollary we get that the complex K_A, constructed in the previous section is a $K(A, 1)$.

Before launching into the fully general proof we give an elementary proof in the case of 3-generators (the 1 and 2-generator cases are trivially lattices).

Theorem 8.1. *For each ordered labeled graph Γ with 3 vertices, the poset P_Γ^W is a lattice.*

Proof. The proof will rely on the fact that the reflection group W_Γ acts nicely on $\mathbb{S}^2$, $\mathbb{R}^2$ or $\mathbb{H}^2$.

(Notice that Noel and I can do the 3-generator cases using geometry, and this should probably be the model for the general case. In particular, we define the "down-set" as the set of reflections below a particular element of height 2. It is either a fixed point or a translation direction. The idea is then that given two distinct elements at height 2, there is at most one reflection which has both bits of information (two fixed points, a fixed point and a normal direction, two normal directions). This completes the proof. One minor glitch is that we need to know that distinct rotations have distinct fixed points and that distinct translations have distinct translation axes. Again we can do this, but it relies on the fact that we're in low-dimensions.) ☐

Theorem 8.2 (Lattice)**.** *For every finite labeled graph Γ, the poset P_Γ is a lattice.*

Proof. First note that by self-duality, it is sufficient to prove that least upper bounds (or greatest lower bounds) exist.

(The proof might be algebraic or geometric. The geometric approach is via min-sets. Coming down from δ each element should increase the size of the min-set by a dimension. In fact it must since each reflection can only increase the min-set by one dimension.)

(Noel and I also have various (re)statements using palindromes and min-sets, which may or may not be useful.)

(to be completed later) ☐

The reason for investigating this is that P lattice should immediately imply that $\tilde{K}$ is contractible and that K is an Eilenberg-Maclane space for A.

Corollary 8.3 (Eilenberg-Maclane)**.** *Let Γ be ordered labeled graph. If Conjecture B holds for Γ, then the complex K_Γ is a finite dimensional Eilenberg-Maclane space for A_Γ. As a consequence A_Γ has finite cohomological dimension and is torsion-free.*

Proof. (Hopefully, by the time this is submitted, we can simply quote that a lattice fundamental domain implies $K(G, 1)$ lemma in some paper by John Meier and I.) ☐

9. Retracting and finite Eilenberg-Maclane spaces

The main goal of this section is to give a deformation retraction argument that Noel and I came up with. The upshot is that if Conjecture B is true for Γ then A_Γ has a finite (not just finite dimensional) Eilenberg-Maclane space.

Theorem 9.1 (Finiteness)**.** *If Conjecture B holds for a finite labeled graph* Γ*, then the Eilenberg-Maclane space* K_Γ *deformation retracts onto a finite subcomplex.*

Proof. (The idea is to "push" from infinity, interatively starting with the top dimensions. We push if and only if the column we are staring at is non-repetative. This is the idea at least. One worry is whether there could exist "slant" infinities.) □

Corollary 9.2. *Every finitely generated Artin group for which Conjecture B holds has a finite Eilenberg-Maclane space.*

10. Positive monoids and linearity

In this section we show how Conjecture C implies that A_Γ is linear.

Theorem 10.1 (Linearity)**.** *Let* Γ *be an ordered labeled graph. If Conjecture C holds for* Γ*, then the Artin group* A_Γ *is linear*

Proof. (The main idea is that Paris' representation is injective on the larger positive monoid (with minimal work – fingers crossed). The linearity then follows by the standard argument.)

(the standard argument might involve knowing that P_Γ is a lattice as well. Then defining a normal form and showing that every word can be written in the form $U^{-1}V$ using the Coxeter element. Does this need lattice or not?) □

11. Combinatorial approach to lattice

Things to do:

1. show that the garside structure for Γ is a quotient of the free graded garside structure $\mathcal{C}$.
2.

12. Final remarks

Approaches to showing the lattice condition:

1. Combinatorial (quotienting a lattice by a group action)
2. Topological (push John's lattice proof in the free case)
3. Coxeter (push M-Noel's/Tom-Colum's proof in the 3-generator/finite-type case to an arbitrary Davis complex)

Connections with

1. non-commutative geometry and free-probabilty
2. outer space.
3. braided categories and quantum groups.

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