WORKED EXAMPLES FOR ARTIN GROUPS OF AFFINE TYPE

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Abstract. The executive summary is that most of the affine Coxeter groups are not lattices. In particular, the infinite families $\widetilde{B}_n$ and $\widetilde{D}_n$ as well as $\widetilde{F}_4$, $\widetilde{E}_6$, $\widetilde{E}_7$, and $\widetilde{E}_8$. The ones which are lattices are $\widetilde{A}_n$, $\widetilde{C}_n$ and $\widetilde{G}_2$ (but the proof for $\widetilde{C}_n$ hasn't been completed). This file is meant to record these calculations in a usable form.

This file contains the lattice calculations for the affine Artin groups. After a general section explaining the methods used, the groups are considered in alphabetical order, with the exception of the affine Coxeter groups of type $\widetilde{A}_n$, $n \geq 2$. These groups are different from the others in that these are the only finite or affine Coxeter groups whose diagrams are not trees. As a result there can be multiple conjugacy classes of Coxeter elements and the techniques used for this type are slightly different. In order to maintain the flow of the narrative, we have postponed our discussion of this type to the end.

1. The techniques

Every affine Coxeter group has a standard set of generators consisting of (s of) α_i, $i = 1, \ldots, n$ which generate a finite Coxeter group and $\widetilde{\alpha}$ which makes it affine.

Introduce the hyperplanes $H_{\alpha,k}$ and the coroots. The elements $s_{\alpha,k}$.

1.1. Coxeter elements.

A *Coxeter element* is a product of the standard reflections in some fixed total ordering of the vertices of the Coxeter diagram. Since many of the standard reflections commute, several different orderings produce the exact same group element. In fact, the crucial information is contained in the ordering of pairs s_i and s_j of standard reflections that are joined by an edge in the Coxeter diagram. We adopt the convention that the edge is oriented from i to j iff s_i occurs before s_j in the factorization.

Conjugacy classes of Coxeter elements using quivers and reflection functors.

Lemma 1.1. *The affine Coxeter groups $\widetilde{A}_n$ have $\lfloor \frac{n}{2} \rfloor$ conjugacy classes of Coxeter elements. Every other affine Coxeter group has only one conjugacy class of Coxeter element.*

Bipartite Coxeter elements.

How to calculate the direction of the Coxeter element.

Lemma 1.2 (Coxeter line). *The direction of the Coxeter line can be found by separating the equation for the highest root into its bipartite parts.*

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Proof. Let W be an affine Coxeter group, let σ be a fundamental chamber for W and let γ be a bipartite Coxeter element. The The red facets of σ intersect in a subsimplex τ_r and the blue facets intersect in a disjoint subsimplex τ_b. If we let t be the shortest line segment connecting τ_r and τ_b, then the direction of t is the Coxeter direction. Moreover, the minimality of t implies that t is perpendicular to the affine span of τ_r and to the affine span of τ_b. Equivalently, the direction of t is contained in the span of the red root directions and it is contained in the span of the blue root directions. A dimension count, together with the fact that the red roots (and the blue roots) are linearly independent, shows that this direction is unique. Finally, consider the equation the relates $\widetilde{\alpha}$ to the other roots and rewrite it so that all of the red roots occur on one side of the equal sign and all of the blue roots occur on the other. In this context, the new root $\widetilde{\alpha}$ is canonically either red or blue. At this point both sides describe the same vector $\vec{v}$. The two descriptions show that $\vec{v}$ is the unique vector that lies in the intersection of the two spans. □

(add an example such as $\widetilde{E}_6$.)

The shortest distance between these two is a line segment which extends to a geodesic under the action of τ_- and τ_+.

Lemma 1.3. $\widetilde{\alpha} \cdot \Delta = 2$.

How to calculate the length of the Coxeter translation.

1.2. **Bowties.** How to calculate which translations occur in the fundamental domain.

Definition 1.4 (Bowties)**.** A bowtie exists in affine poset if and only if there is a tetrahedron of roots such that with a pair of oppositve

How to check for bowties.

2. $\widetilde{A}_n$ TYPE

Each conjugacy class leads to a different partially ordered set, one of which is a lattice and all the others are not. This is because there is a $K_{p,q}$ (with $p+q=n$) in the translation group.

3. $\widetilde{B}_n$ TYPE

simple root	coordiates
α_1	$\epsilon_1 - \epsilon_2$
α_2	$\epsilon_2 - \epsilon_3$
$\vdots$	$\vdots$
α_{n-1}	$\epsilon_{n-1} - \epsilon_n$
α_n	ϵ_n
$\widetilde{\alpha}$	$\epsilon_1 + \epsilon_2$

FIGURE 1. The simple roots for type $\widetilde{B}_n$.

The direction of Δ is $(0, 2, -2, 2, -2, \ldots, \pm 2)$. Translation group consists of the root $\pm\epsilon_1 + (-1)^k \epsilon_k$. There are definitely bowties.

4. $\widetilde{C}_n$ TYPE

This one is a lattice (we think).

5. $\widetilde{D}_n$ TYPE

Roots
Delta
Translation group
Bowties

6. $\widetilde{E}_6$ TYPE

Definition 6.1 (Root conventions)**.** Our convention for listing the roots is as follows. The numbers before the slash are $+$, the numbers after are $-$, the missing numbers are 0. If there are only two numbers listed they are ± 1. If there are eight numbers listed they are $\pm\frac{1}{2}$. Each row forms a K_5. The top half and the bottom half of each column forms a K_3 and the each entry in the top half of the column forms a K_2 with the corresponding in entry in the bottom half.

This type of convention can be used for all of the root systems.

The roots in E_6 are ... and the Coxeter direction is $\Delta = (0, 4, -2, 2, 0, 0, 0, 0, 0)$.

14/	45/	123458/67
4/5	4/1	2348/1567
123467/58	234567/18	23/
1/3	5/3	1258/3467
/35	/13	28/134567
1267/3458	2567/1348	2/4

FIGURE 2. The 18 good roots in $\widetilde{E}_6$ organized according to its product structure.

7. $\widetilde{E}_7$ TYPE

The roots in E_7 are ... and the Coxeter direction is $\Delta = (-1, 5, -3, 3, -1, 1, -1, 1)$. The 24 good translation directions are listed in Figure 3.

1/3	5/3	/36	1258/3467
14/	45/	4/6	123458/67
1468/2347	4568/1237	48/123567	8/7
1267/3458	2567/1348	27/134568	2/4
123467/58	234567/18	2347/1568	23/
6/5	6/1	/15	2368/1457

FIGURE 3. The 24 good roots in $\widetilde{E}_7$ organized according to its product structure.

8. $\widetilde{E}_8$ type

The roots in E_8 are the 56 roots of the form $\pm\epsilon_i \pm \epsilon_j$ plus the 184 roots of the form $\frac{1}{2}\sum_{i\in[8]} \pm\epsilon_i$. The Coxeter direction is $\Delta = (-1, 7, -5, 5, -3, 3, 1, 1)$. The 30 good translations directions are listed in Figure 4.

/57 6/7 2368/1457	/58 6/8 2367/1458	1/5 16/ 145678/23	1234/5678 123456/78 23/	12/345678 1256/3478 2/4
48/123567 4568/1237 8/1	47/123568 4567/1238 7/1	1478/2356 145678/23 78/	4/6 45/ 234578/16	/36 5/3 2578/1346

Figure 4. The 30 good roots in $\widetilde{E}_8$ organized according to its product structure.

Translation group

9. $\widetilde{F}_4$ type

Roots
Delta
Translation group

10. $\widetilde{G}_2$ type

This one is trivial a lattice because it has only 3 generators. Nevertheless, it is a good exercise to complete the above list by going through the same set of particulars.
Roots
Delta
Translation group

11. Creating a lattice for Exo($\widetilde{B}_3$)

This is our first (and most detailed) example of a lattice completion.

In the exoskeleton of $\widetilde{B}_3$ there are 18 elements and 8 generators. Traditionally we called these a_i, $i \in [4]$, for the 4 vertical reflections and x_i, $i \in [4]$, for the 4 allowable translations, but I want to introduce a slightly new notation. Let $\{1, 2, 3, 4\}$ be the 4 fixed lines below Δ, cyclically labeled. Then there are 4 reflections $a_{12}, a_{23}, a_{34}, a_{41}$ labeled by the two lines they contain. Notice that the subscripts are adjacent and that they should be considered unordered. The 4 translations are $x_1^3, x_3^1, x_2^4, x_4^2$ labeled by the two lines they connect. Notice that this time the order matters.

The presentation derived from $\textsc{Exo}(\widetilde{B}_3)$ has the following basic relations.

$$\begin{array}{lcl} a_{12}\cdot a_{23} &=& a_{23}\cdot a_{12} \\ a_{23}\cdot a_{34} &=& a_{34}\cdot a_{23} \\ a_{34}\cdot a_{41} &=& a_{41}\cdot a_{34} \\ a_{41}\cdot a_{12} &=& a_{12}\cdot a_{41} \end{array} \qquad \begin{array}{l} a_{12}\cdot x_2^4 = x_2^4\cdot a_{34} = a_{34}\cdot x_3^1 = x_3^1\cdot a_{12} \\ a_{23}\cdot x_3^1 = x_3^1\cdot a_{41} = a_{41}\cdot x_4^2 = x_4^2\cdot a_{23} \\ a_{34}\cdot x_4^2 = x_4^2\cdot a_{12} = a_{12}\cdot x_1^3 = x_1^3\cdot a_{34} \\ a_{41}\cdot x_1^3 = x_1^3\cdot a_{23} = a_{23}\cdot x_2^4 = x_2^4\cdot a_{41} \end{array}$$

The way to remember these relations is that two a's commute iff they share a subscript and there is a relation between an a and x if the appropriate x-number is shared with a. Here, appropriate means the subscript when the a is before the x and the superscript when the a is after the x.

Lemma 11.1. *The factorizations of Δ in this notation are of the form $x\cdot a\cdot a$ or $a\cdot x\cdot a$ or $a\cdot a\cdot x$. In all three cases, the a's before the x need to contain the x subscript and the a's after the x need to contain the x subscript. Finally, if there is an a before and after the x, only three numbers should occur as subscripts and superscripts. Every triple satisfying these conditions is a factorization of Δ.*

For example, $x_1^3\cdot a_{34}\cdot a_{23}$ is one factorization of Δ and $a_{12}\cdot x_1^3\cdot a_{23}$ is another, but $a_{12}\cdot x_1^3\cdot a_{34}$ is not.

11.1. **The lattice completion.** Once we try and equivariantly complete the poset we add in a number of new generators and relations. [Our first attempt is contained in the "old stuff" section below. Here's a revised version.] Let z_{34}^{12} be the new edge added that is perpendicular to the 12-plane and the 34-plane and in the direction from the 34-plane to the 12-plane. Using this as our guide we find that the z's commute and that z_{34}^{12} acts as follows with the a's. We have the existing relations and the factorizations.

$$\begin{array}{lcl} a_{12}\cdot a_{23} &=& a_{23}\cdot a_{12} \\ a_{23}\cdot a_{34} &=& a_{34}\cdot a_{23} \\ a_{34}\cdot a_{41} &=& a_{41}\cdot a_{34} \\ a_{41}\cdot a_{12} &=& a_{12}\cdot a_{41} \end{array} \qquad \begin{array}{l} z_{12}\cdot z^{23} = z_{41}\cdot z^{34}\ (= x_1^3) \\ z_{23}\cdot z^{34} = z_{12}\cdot z^{41}\ (= x_2^4) \\ z_{34}\cdot z^{41} = z_{23}\cdot z^{12}\ (= x_3^1) \\ z_{41}\cdot z^{12} = z_{34}\cdot z^{23}\ (= x_4^2) \end{array}$$

There are 8 new commutations.

$$\begin{array}{l} a_{12}\cdot z_{23}^{41} = z_{23}^{41}\cdot a_{12} \\ a_{23}\cdot z_{34}^{12} = z_{34}^{12}\cdot a_{23} \\ a_{34}\cdot z_{41}^{23} = z_{41}^{23}\cdot a_{34} \\ a_{41}\cdot z_{12}^{34} = z_{12}^{34}\cdot a_{41} \end{array} \qquad \begin{array}{l} a_{12}\cdot z_{41}^{23} = z_{41}^{23}\cdot a_{12} \\ a_{23}\cdot z_{12}^{34} = z_{12}^{34}\cdot a_{23} \\ a_{34}\cdot z_{23}^{41} = z_{23}^{41}\cdot a_{34} \\ a_{41}\cdot z_{34}^{12} = z_{34}^{12}\cdot a_{41} \end{array}$$

And then there are the interesting conjugations.

$$\begin{array}{l} a_{12}\cdot z_{12}^{34} = z_{12}^{34}\cdot a_{34} = a_{34}\cdot z_{34}^{12} = z_{34}^{12}\cdot a_{12} \\ a_{23}\cdot z_{23}^{41} = z_{23}^{41}\cdot a_{41} = a_{41}\cdot z_{41}^{23} = z_{41}^{23}\cdot a_{23} \end{array}$$

11.2. **Old stuff.** in that ends where x_3^1 and x_4^2 meet. All of the other $z^{i,i+1}$ and $z_{i,i+1}$ are defined similarly. All of the x's can be eliminated as a consequence of the new relations. Here are the obvious ones from the diagrams.

$$\begin{array}{l} x_1^3 = z_{12}\cdot z^{23} = z_{41}\cdot z^{34} \\ x_2^4 = z_{23}\cdot z^{34} = z_{12}\cdot z^{41} \\ x_3^1 = z_{34}\cdot z^{41} = z_{23}\cdot z^{12} \\ x_4^2 = z_{41}\cdot z^{12} = z_{34}\cdot z^{23} \end{array} \qquad \begin{array}{l} z^{12}\cdot a_{12} = z^{34}\cdot a_{34} \\ z^{23}\cdot a_{23} = z^{41}\cdot a_{41} \\ a_{12}\cdot z_{12} = a_{34}\cdot z_{34} \\ a_{23}\cdot z_{23} = a_{41}\cdot z_{41} \end{array}$$

The rules are that lower and upper z's multiply when they have exactly one number in common whereas the z's and a's multiply when upper versus lower matches and both numbers match. Finally, because of the way that cuts nest, there are also extra arrows among these new vertices giving relations involving new edges b and c.

$$\begin{array}{ll} a_{i,i+1} \cdot z_{12} = z_{12} \cdot b_{i,i+1} & b_{i,i+1} \cdot z^{23} = z^{23} \cdot a_{i+2,i+3} \\ a_{i,i+1} \cdot z_{34} = z_{34} \cdot b_{i,i+1} & b_{i,i+1} \cdot z^{41} = z^{41} \cdot a_{i+2,i+3} \\ a_{i,i+1} \cdot z_{23} = z_{23} \cdot c_{i,i+1} & c_{i,i+1} \cdot z^{12} = z^{12} \cdot a_{i+2,i+3} \\ a_{i,i+1} \cdot z_{41} = z_{41} \cdot c_{i,i+1} & c_{i,i+1} \cdot z^{34} = z^{34} \cdot a_{i+2,i+3} \end{array}$$

Next, the b's and the c's satisfy commutation relations similar to the a's.

$$\begin{array}{lllllllll} a_{12} \cdot a_{23} & = & a_{23} \cdot a_{12} & b_{12} \cdot b_{23} & = & b_{23} \cdot b_{12} & c_{12} \cdot c_{23} & = & c_{23} \cdot c_{12} \\ a_{23} \cdot a_{34} & = & a_{34} \cdot a_{23} & b_{23} \cdot b_{34} & = & b_{34} \cdot b_{23} & c_{23} \cdot c_{34} & = & c_{34} \cdot c_{23} \\ a_{34} \cdot a_{41} & = & a_{41} \cdot a_{34} & b_{34} \cdot b_{41} & = & b_{41} \cdot b_{34} & c_{34} \cdot c_{41} & = & c_{41} \cdot c_{34} \\ a_{41} \cdot a_{12} & = & a_{12} \cdot a_{41} & b_{41} \cdot b_{12} & = & b_{12} \cdot b_{41} & c_{41} \cdot c_{12} & = & c_{12} \cdot c_{41} \end{array}$$

And there are a few more relations visible as well. (I should think these through carefully in the morning).

$$\begin{array}{lll} z_{12} \cdot b_{34} & = & z_{34} \cdot b_{12} \\ z_{23} \cdot c_{41} & = & z_{41} \cdot c_{23} \\ c_{12} \cdot z^{34} & = & c_{34} \cdot z^{12} \\ b_{23} \cdot z^{41} & = & b_{41} \cdot z^{23} \end{array}$$

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