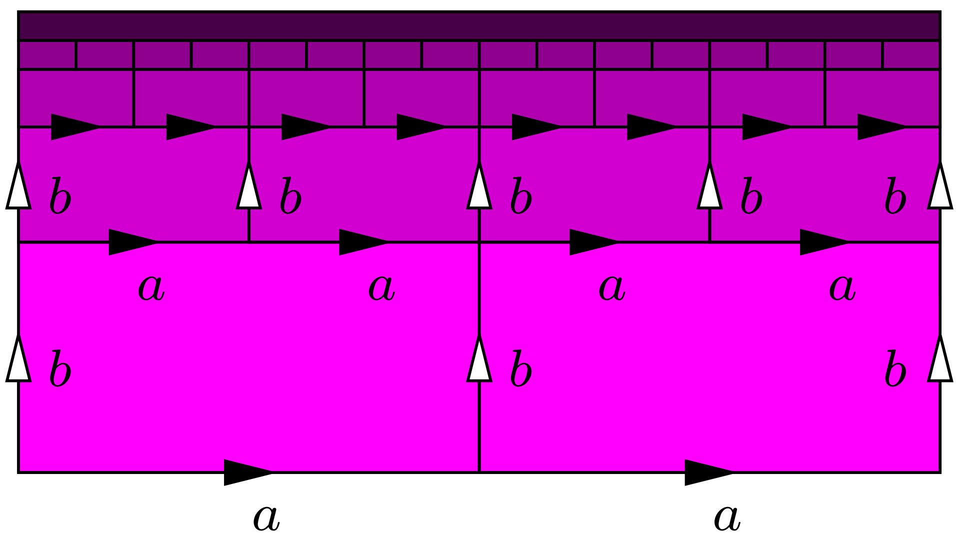
Constructing non-positively curved spaces and groups

Day 4: Combinatorial notions of curvature



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Outline

I. Angles in Polytopes

II. Combinatorial Gauss-Bonnet

III. Conformally CAT(0)

IV. One-relator groups

I. Angles in Polytopes

Let F be a face of a polytope P.

• The normalized *internal angle* $\alpha(F, P)$ is the proportion of unit vectors perpendicular to F which point into P (i.e. the measure of this set of vectors divided by the measure of the sphere of the appropriate dimension).

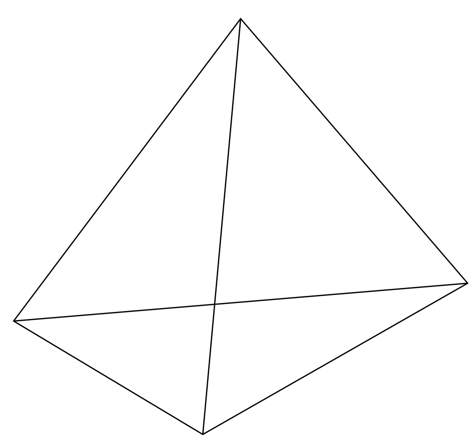
• The normalized *external angle* $\beta(F, P)$ is the proportion of unit vectors perpendicular to F so that there is a hyperplanes with this unit normal which contains F and the rest of P is on the other side.

Thm: $\sum_{v \in P} \beta(P, v) = 1.$

Angle Sums

The sum of the internal angles in a triangle is π, but the sum of the dihedral angles in a tetrahedron can vary.

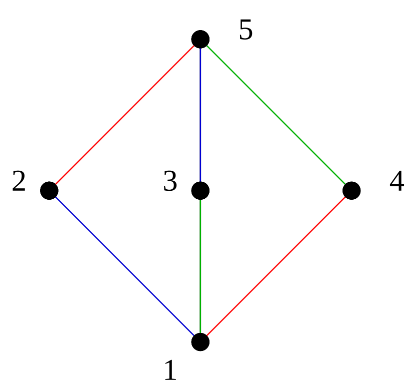
There are relations between the various internal and external angles in a Euclidean polytope but we will need a digression into combinatorics in order to state the relationship properly.



Posets and Incidence algebras

Let P be a finite poset with elements labeled by $[n]$. The set of $n \times n$ matrices with $a_{ij} \neq 0$ only when $i \leq_P j$ is called the *incidence algebra of P*, $I(P)$.

For any finite poset P there is a numbering of its elements which is consistent with its order. In this ordering, the incidence algebra is a set of upper triangular matrices.



$$\zeta_P = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

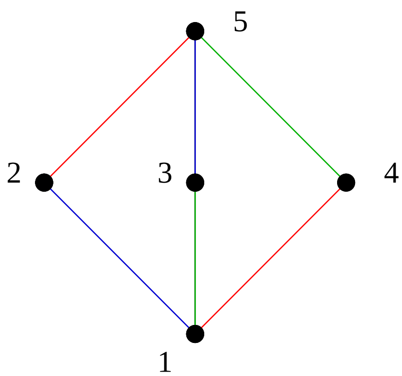
Delta, Zeta and Möbius functions

Rem: The elements of $I(P)$ can also be thought of as functions from $P \times P \rightarrow \mathbb{R}$.

The identity matrix is the *delta function* where $\delta(x, y) = 1$ iff $x = y$.

The *zeta function* is the function $\zeta(x, y) = 1$ if $x \leq_P y$ and 0 otherwise (i.e. 1's wherever possible).

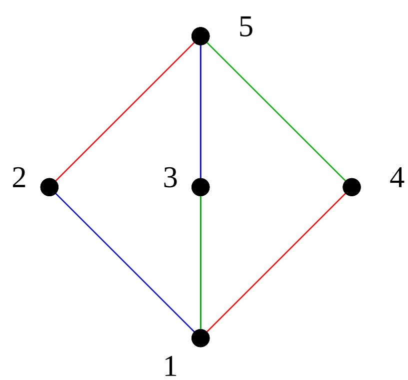
The *möbius function* is the matrix inverse of ζ. Note that $\mu\zeta = \zeta\mu = \delta$.



$$\mu_P = \begin{bmatrix} 1 & -1 & -1 & -1 & 2 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Möbius functions and Euler characteristics

Let P be a finite poset and let $\widehat{P}$ be the same poset with the addition of a new minimum element $\hat{0}$ and a new maximum element $\hat{1}$. The value of the möbius function on the interval $(\hat{0}, \hat{1})$ is the reduced Euler characteristic of the geometric realization of the poset P.

$\widehat{P} =$ 

$\tilde{\chi}(P) = 2$

In this example the realization of P is 3 discrete points.

Digression on $\ell^{(2)}$ Betti numbers

Following the type of philosophy espoused in Wolfgang Lück's talks, John Meier and I recently calculated the $\ell^{(2)}$ Betti numbers of the pure symmetric automorphism groups with very few calculations.

We used

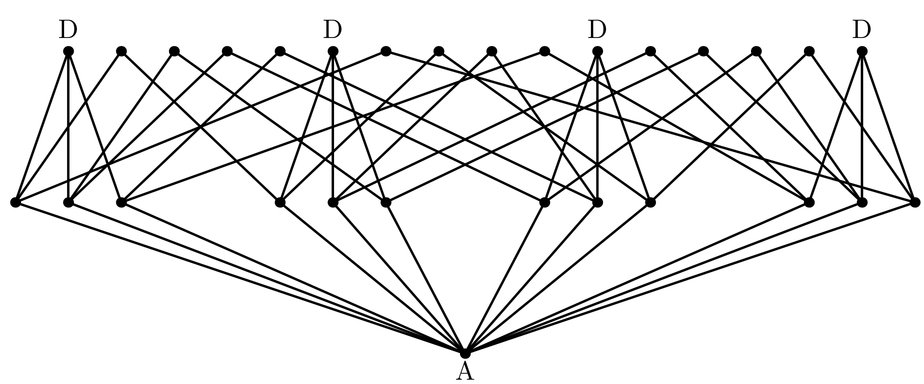
• a spectral sequence to show that all but the top Betti number was 0,

• the final Betti number must be the Euler characteristic of the fundamental domain,

• which comes from the möbius function,

• which we computed using techniques from enumerative combinatorics.

$HT_4 =$



Incidence algebras for Polytopes

The faces of a Euclidean polytope under inclusion is its *face lattice*. Traditionally $\hat{0} = \emptyset$ is added so that the result is a lattice in the combinatorial sense.

The set of all internal (external) angles forms an element of the incidence algebra of the face lattice, α (β).

Rem: The notion of internal and external angle needs to be extended so that $\alpha(\hat{0}, F)$ and $\beta(\hat{0}, F)$ have values, and there are many natural ways to do this.

Möbius functions for Polytopes

Lem: The möbius function of the face lattice of a polytope is $\mu(F,G) = (-1)^{\dim G - \dim F}$.

Proof: The geometric realization of the portion of the face lattice between F and G is a sphere.

Def: Let $\bar{\alpha}(F,G) = \mu(F,G)\alpha(F,G)$, [Hadamard product] (i.e. $\bar{\alpha}$ is a *signed* normalized internal angle.

Thm(Sommerville) $\mu\alpha = \bar{\alpha}$ i.e.

$$\sum_{F \le G \le H} \mu(F,G)\alpha(G,H) = \mu(F,H)\alpha(F,H)$$

Equations for angles

The most interesting of angle identity is the one discovered by Peter McMullen.

Thm(McMullen) $\alpha\beta = \zeta$, i.e.

$$\sum_{F \leq G \leq H} \alpha(F,G)\beta(G,H) = \zeta(F,H)$$

Proof Idea:

- Look at (a polytopal cone) $\times$ (its dual cone)

- Integrate $f(\vec{x}) = \exp(-||\vec{x}||^2)$ over this $\mathbb{R}^{2n}$ in two different ways.

Cor: $\mu\alpha\beta = \bar{\alpha}\beta = \delta$.

Curvature in PE complexes

Following Cheeger-Müller-Schrader (and Charney-Davis), if X is a PE complex

$$\begin{aligned}\chi(X) &= \sum_P (-1)^{\dim P} \\ &= \sum_P \sum_{v\in P} (-1)^{\dim P}\beta(v,P) \\ &= \sum_v \sum_{P\ni v} (-1)^{\dim P}\beta(v,P) \\ &= \sum_v \kappa(v)\end{aligned}$$

where $\kappa(v) := \sum_{P\ni v} (-1)^{\dim P}\beta(v,P)$.

Rem 1: $\kappa(v)$ is similar to (but not) a signed version of β.

Rem 2: The first step is really just replacing δ with $\bar{\alpha}\beta$ in a very precise sense.

II. Combinatorial Gauss-Bonnet

An *angled 2-complex* is one where we assign normalized external angles $\beta(v,f)$ for each vertex v in a face f.

Define $\kappa(v)$ as above. Define $\kappa(f)$ as a correction term which measures how far the external vertex angles are from 1.

$$\kappa(f) = 1 - \sum_{v \in f} \beta(v,f)$$

Thm(Gersten,Ballmann-Buyalo,M-Wise)
If X is an angled 2-complex, then

$$\sum_{v} \kappa(v) + \sum_{f} \kappa(f) = \chi(X)$$

Rem: In all these papers the sum was $2\pi\chi(X)$ since the angles were not normalized. As we have seen normalization is crucial for the equations in higher dimensions.

Combinatorial Gauss-Bonnet in higher dimensions

The formula $\sum_{v \in P} \beta(v, P) = 1$ is a consequence of McMullen's theorem under one extension of α and β to the intervals $(\hat{0}, F)$.

Similarly, the combinatorial Gauss-Bonnet Theorem on the previous slide comes from reversing the order of summation for another factorization of the zeta function.

General CGB "Thm" Given any factorization $\alpha\beta = \zeta$, reversing the order of summation gives a combinatorial Gauss-Bonnet type formula.

Rem 1: Only factorizations which produce lots of 0s will be of much use, but there is room to explore.

Rem 2: The Regge calculus should also fit into this framework.

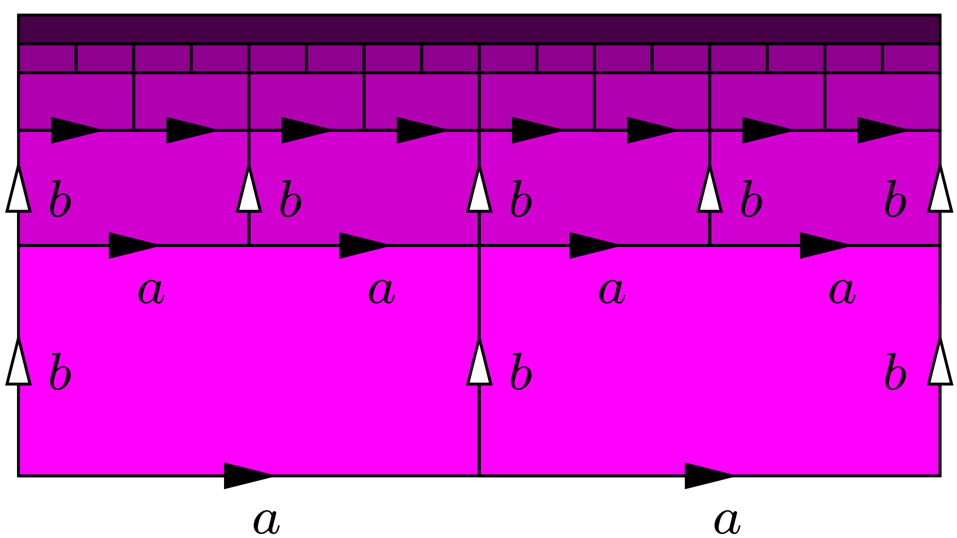
III. Conformal CAT(0) structures

A 2-complex X with an angle assigned to each corner is an *angled 2-complex*.

If the vertex links are CAT(1), then X is called *conformally* CAT(0).

Thm(Corson): Conformally CAT(0) 2-complexes are aspherical.

Example: The Baumslag-Solitar groups are conformally CAT(0) - even though they are not CAT(0), except in the obvious cases.



Sectional curvature

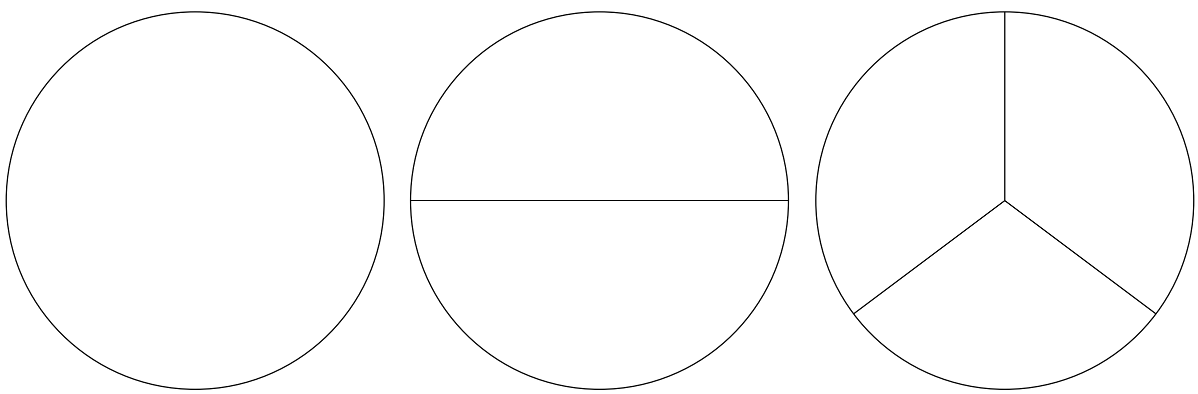
Def: Let X be an angled 2-complex. If every connected, 2-connected subgraph of each vertex link is CAT(1), then X has *non-positive sectional curvature*.

Thm(Wise) If X is an angled 2-complex with non-positive sectional curvature, then $\pi_1 X$ is coherent.

Rem: Using Howie towers, these are the key types of sublinks that need to be considered.

Special polyhedra

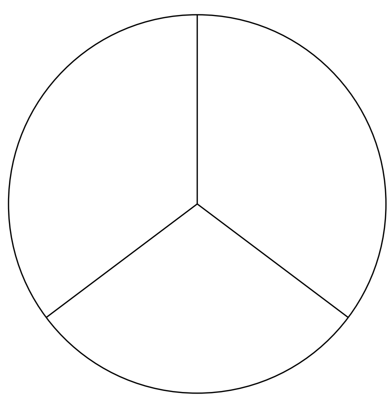
Def: A 2-complex is called a *special polyhedron* if the link of every point is either a circle, a theta graph, or the complete graph on 4 vertices. These points define the intrinsic 2-, 1- and 0-skeleta of X.



Conformal CAT(0) structures and Special polyhedra

Lem: If X is an angled 2-dimensional special polyhedron, then X is conformally CAT(0) if and only if X has non-positive sectional curvature.

Pf: The only subgraphs to check are triangles, and whole graph.



Cor: If X is a 2-dimensional special polyhedron with a conformal CAT(0) structure, then $\pi_1 X$ is coherent.

IV. One-relator groups

Conj A: Every one-relator group is coherent.

Conj B: Every one-relator group is the fundamental group of a 2-dimensional special polyhedron with a conformal CAT(0) structure.

Rem 1: Conjecture B implies Conjecture A, and it would help explain why one-relator groups tend to "act like" non-positively curved groups.

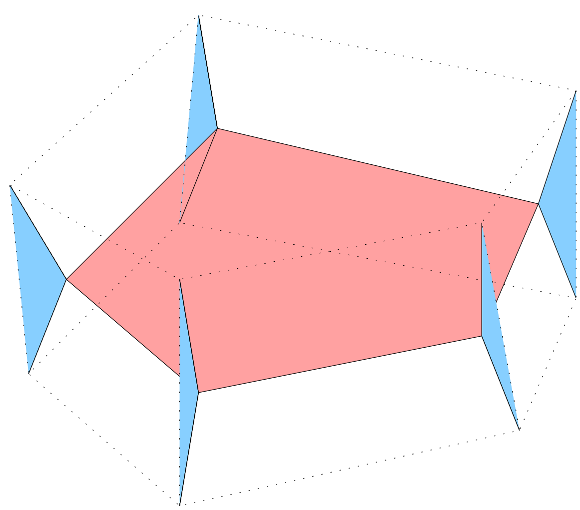
Rem 2: For Conjecture A it is sufficient to prove Conjecture B for 2-generator one-relator groups since every one-relator group is a subgroup of a 2-generator one-relator group. Moreover, the inequalities are tight (and become equations) in this case.

Special polyhedra for one-relator groups

Def: If x is a point in $X^{(2)}$ such that $X - x$ deformation retracts onto a graph, then x is a *puncture point*.

Rem: If X has a puncture point then $\pi_1 X$ is a one-relator group.

Thm(N.Brady-M) If X is the presentation 2-complex for a one-relator group, then X is simply-homotopy equivalent to a 2-dimensional special polyhedron Y with a puncture point. In addition, Y can be chosen so that it has no monogons, bigons, or untwisted triangles.



Additional remarks

The puncture point and $\chi(X) = 0$ allow you to remove most portions of the 2-skeleton which are not discs.

The game is to use the flexibility in the special polyhedron construction to manipulate the linear system so that it has a solution. Since this system has $3n$ variables and $2n$ equations, our odds are good in general – we only need to avoid contradictions.

The first several examples we tried by hand produced conformal CAT(0) structures, even when we proceeded "randomly".

A computer program to check all the one-relator groups out to a modest size is high on my to-do-list.