Recovering knot polynomials from graphs

Sanjay Kumar & Melody Molander

Prerequisites

For the project, all of the necessary background information will be provided prior to working on the research project; however, it is highly recommended to be acquainted with the following prerequisites:

- Undergraduate level linear algebra
- Although not necessary, a background in programming would be useful.
- Familiarity with writing mathematical proofs
- Strong problem solving skills
- Motivation and determination

Even though much of the language in the project is technical, you should not feel discouraged to apply. The following project is topological in nature, and many of the topics have an intuitive interpretation as a result.

Introduction

In the field of topology, mathematical knots are one of the central objects of study. Although their construction is relatively straightforward, knots have many surprising connections with different fields such as physics, biology, and computer science.

Rigorously, *knots* are embeddings of circles into 3-dimensional spaces; however, they can be thought of intuitively as a rope with their ends tied together. In a more general setting, we call a finite collection of knots a *link*. A natural direction in the field is whether we can classify different knots. In particular, we define two knots to be equivalent if you can transform one into another by manipulating the knot without cutting or passing through itself. This process is known as an *ambient isotopy*, and we say that two knots are *distinct* if there does not exist an ambient isotopy between them. This leads to the fundamental question on whether we can determine if two knots are distinct.

If you approach this directly, the question becomes very difficult to answer. In general, how does one determine whether there does not exist an ambient isotopy between two knots? For example, the two knots may not appear equivalent; however, there may be an obscure transformation between the two. A method to resolve this issue is through the use of *knot invariants*. Roughly, a knot invariant is an assignment of a "quantity" to a given knot. If two knots are assigned different "quantities", then they must be distinct.

A common type of knot invariant is a *knot polynomial* which algorithmically assigns a polynomial to each knot. The first example of such was the Alexander polynomial which was constructed in the 1920's; however, within the last 40 years, due to the discovery of the Jones polynomial as well as the foundational work of Witten, Reshetikhin, and Turaev, a wealth of new polynomial invariants have been introduced. One of the key components in their construction was the polynomials' surprising connection to quantum mechanics.

In addition to distinguishing knots, the underlying meaning of the invariants have become an interesting area of study in itself. For example, there are many conjectures which state that the polynomials contain intrinsic geometric data of the 3-dimensional space. These conjectures are often difficult to approach due to the complexity involved in calculating the invariant. Because of this, it is useful to find more efficient methods in obtaining the polynomial.

Description of the project

The overarching goal of the project is to obtain knot polynomials using different techniques with the possibility of simplifying the overall calculation. One such method is through the use of *graph theory* which is the study of vertices connected by edges. In an analogous way to knot theory, one can also define polynomial invariants of graphs.

As our motivation, it has been shown by various authors such as Cohen, Dasbach, Jaeger, and Russell that there are pairings between some knots and graphs such that certain knot polynomials can be recovered from the corresponding graph polynomial. As a specific example, it has been shown by Cohen that the Jones polynomial of a family of knots known as the *pretzel knots* can be calculated quickly using an associated graph invariant.

In the summer of 2022, we will be working on finding other knot polynomials which can be obtained from graph polynomials for certain families of knots as well as other possible problems which arise.