Week 10 6PM

Tuesday, June 2, 2020 6:01 PM

* Last Discussion!
* Finals OH
* Course Evals

We can compute triple integrals similar to double integrals except now our regions are 3D.

Our bounds for integration...

$f(y,z) \le \underline{x} \le g(y,z)$ — 2 variables $\quad f(x,y) \le z \le g(x,y)$

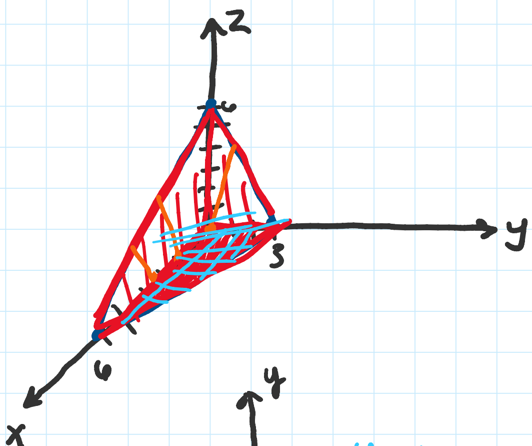
$h(z) \le \underline{y} \le i(z)$ — 1 variable $\quad h(y) \le x \le i(y)$

$a \le \underline{z} \le b$ — 0 variables $\quad a \le y \le b$

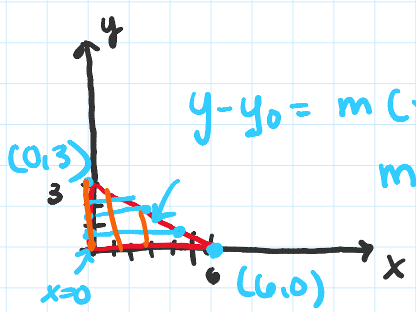
Ex Evaluate $\boxed{\iiint_W 2y\,dV}$ where W is the solid in the 1st octant bounded by the plane $\boxed{x+2y+z=6}$ $\quad z = 6-x-2y$

$x \ge 0,\ y \ge 0,\ z \ge 0$

$$\begin{cases} x=0,\ y=0 & z=6 \\ x=0,\ z=0 & 2y=6 \quad y=3 \\ y=0,\ z=0 & x=6 \end{cases}$$



$$\begin{cases} 0 \le z \le 6-x-2y \\ 0 \le x \le -2y+6 \\ 0 \le y \le 3 \end{cases}$$



$y - y_0 = m(x - x_0)$

$m = \frac{0-3}{6-0} = \frac{-3}{6} = -\frac{1}{2}$

$y - 0 = -\frac{1}{2}(x-6)$

$y = -\frac{1}{2}x + 3$

$y - 3 = -\frac{1}{2}x$

$$y = -\frac{1}{2}x + 3$$

$$y - 3 = -\frac{1}{2}x$$

$$-2y + 6 = x$$

$$\int_{y=0}^{y=3} \int_{x=0}^{x=-2y+6} \left(\int_{z=0}^{z=6-x-2y} 2y \right) dz\, dx\, dy$$

$$= \int_{y=0}^{y=3} \int_{x=0}^{x=-2y+6} 2yz \Big|_{z=0}^{z=6-x-2y} dx\, dy$$

$$= \int_{y=0}^{y=3} \int_{x=0}^{x=-2y+6} 2y(6-x-2y)\, dx\, dy$$

$$= \int_{y=0}^{y=3} \int_{x=0}^{x=-2y+6} (12y - 2xy - 4y^2)\, dx\, dy$$

$$= \int_{y=0}^{y=3} 12yx - x^2y - 4y^2x \Big|_{x=0}^{x=-2y+6} dy$$

$$= \int_{y=0}^{y=3} \left[12y(-2y+6) - (-2y+6)^2 y - 4y^2(-2y+6) \right] dy$$

$$= \int_{y=0}^{y=3} \left(-24y^2 + 72y - (4y^2 - 24y + 36)y + 8y^3 - 24y^2 \right) dy$$

$$= \int_{y=0}^{y=3} \left(\cancel{-24y^2} + \underline{72y} - \underline{4y^3} + \cancel{24y^2} \underline{-36y} + \underline{8y^3} \underline{-24y^2} \right) dy$$

$$= \int_{y=0}^{y=3} (4y^3 - 24y^2 + 36y)\, dy$$

$$= y^4 - 8y^3 + 18y^2 \Big|_{y=0}^{y=3}$$

$$= 3^4 - 8 \cdot 3^3 + 18 \cdot 3^2$$

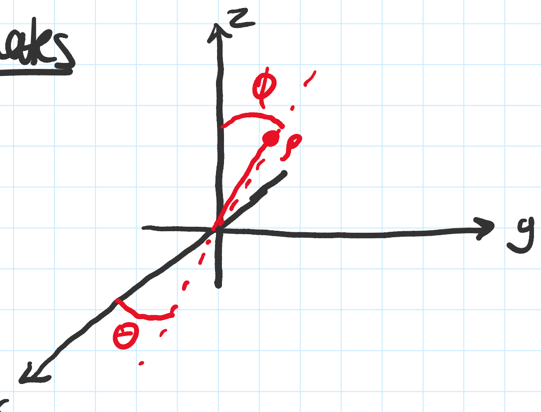
$$= 27$$

Change of Variables

$$\iiint_W f\,dV = \iiint_{W^*} f(x(u,v,w), y(u,v,w), z(u,v,w)) \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| dV^*$$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v & \partial x/\partial w \\ \partial y/\partial u & \partial y/\partial v & \partial y/\partial w \\ \partial z/\partial u & \partial z/\partial v & \partial z/\partial w \end{vmatrix}$$

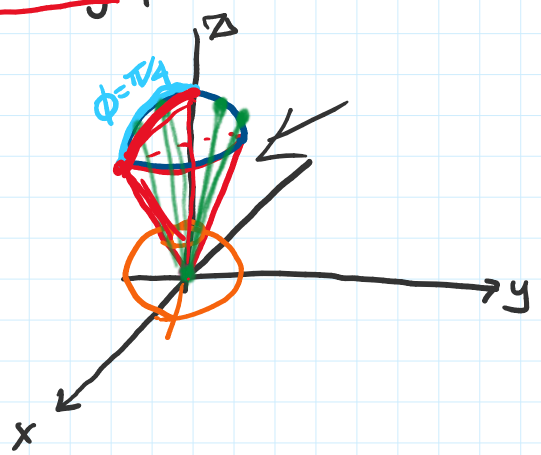
Spherical Coordinates



$x = \rho\sin\phi\cos\theta$ $y = \rho\sin\phi\sin\theta$ $\boxed{z = \rho\cos\phi}$

$\boxed{x^2 + y^2 + z^2 = \rho^2}$

Ex. Find the volume of the solid region that lies inside the sphere $\boxed{x^2+y^2+z^2 = z}$ & above the cone $z^2 = x^2+y^2$, $z \geq 0$ shown below:



$0 \leq \theta \leq 2\pi$

$0 \leq \phi \leq \pi/4$

$0 \leq \rho \leq \cos\phi$

$x^2+y^2+z^2 = z$

$\rho^2 = \rho\cos\phi$ (divide by ρ)

$\rho = \cos\phi$

$x = \rho\sin\phi\cos\theta$ $y = \rho\sin\phi\sin\theta$ $z = \rho\cos\phi$

$\frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)} = \begin{vmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \end{vmatrix}$

$$\frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)} = \begin{vmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{vmatrix}$$

$$= \sin\phi\cos\theta \begin{vmatrix} \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ -\rho\sin\phi & 0 \end{vmatrix} - \rho\cos\phi\cos\theta \begin{vmatrix} \sin\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & 0 \end{vmatrix}$$

$$+ -\rho\sin\phi\sin\theta \begin{vmatrix} \sin\phi\sin\theta & \rho\cos\phi\sin\theta \\ \cos\phi & -\rho\sin\phi \end{vmatrix}$$

$$= \rho^2\sin\phi$$

$$\iiint_W 1 \cdot dV \qquad f = 1$$

$$= \int_{\theta=0}^{\theta=2\pi}\int_{\phi=0}^{\phi=\pi/4}\int_{\rho=0}^{\rho=\cos\phi} 1 \cdot |\rho^2\sin\phi|\, d\rho\, d\phi\, d\theta$$

$$= \int_{\theta=0}^{\theta=2\pi}\int_{\phi=0}^{\phi=\pi/4}\int_{\rho=0}^{\rho=\cos\phi} \rho^2|\sin\phi|\, d\rho\, d\phi\, d\theta$$

$$= \int_{\theta=0}^{\theta=2\pi}\int_{\phi=0}^{\phi=\pi/4} \frac{1}{3}\rho^3|\sin\phi| \Big|_{\rho=0}^{\rho=\cos\phi} d\phi\, d\theta$$

$$= \int_{\theta=0}^{\theta=2\pi}\int_{\phi=0}^{\phi=\pi/4} \frac{1}{3}(\cos\phi)^3|\sin\phi|\, d\phi\, d\theta$$

$$= \int_{\theta=0}^{\theta=2\pi}\int_{\phi=0}^{\phi=\pi/4} \frac{1}{3}\cos^3\phi \cdot \sin\phi\, d\phi\, d\theta$$

$$= \int_{\theta=0}^{\theta=2\pi} \boxed{\frac{1}{3}\cdot\frac{1}{4}} - \cos^4\phi \Big|_{\phi=0}^{\phi=\pi/4} d\theta$$

$$= \boxed{-\frac{1}{12}} \int^{\theta=2\pi} (\cos^4(\pi/4) - \cos^4(0))\, d\theta$$

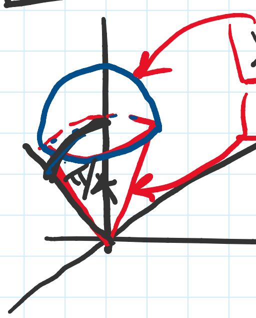
$$= \boxed{-\frac{1}{12}} \int_{\theta=0}^{\theta=2\pi} \left(\cos^4(\pi/4) - \cos^4(0)\right) d\theta$$

$$= -\frac{1}{12} \int_{\theta=0}^{\theta=2\pi} \left(\frac{1}{4} - 1\right) d\theta$$

$$= -\frac{1}{12} \int_{\theta=0}^{\theta=2\pi} -\frac{3}{4} \, d\theta$$

$$= -\frac{1}{12} \cdot -\frac{3}{4} \theta \Big|_0^{2\pi}$$

$$= \boxed{\frac{\pi}{8}}$$



$\boxed{x^2+y^2+z^2 = z} \leftrightarrow \rho^2 = \rho\cos\phi$

$\boxed{z^2 = x^2+y^2} \leftrightarrow \rho^2\cos^2\phi = \rho^2\sin^2\phi\cos^2\theta + \rho^2\sin^2\phi\sin^2\theta$