Math Lab M-F 9-10, 11-4, 6-8 PDT

ucsb.zoom.us/my/mathlab

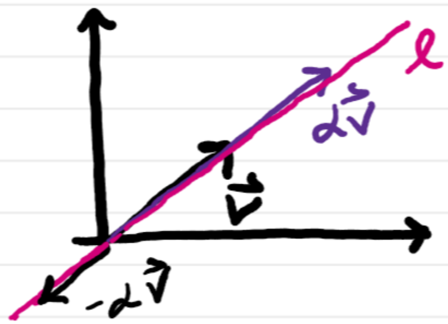
My Math Lab hours : F 12-2

My OH: M 4:30-5:30

Notes & Videos

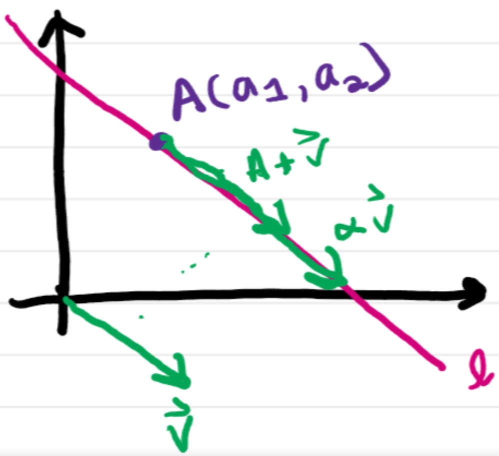
web.math.ucsb.edu/~melodymolander/math6a_spring2020.html

Last time, if $\alpha > 0$ then $\vec{v}$ & $\alpha\vec{v}$ have the same direction



$$\underbrace{\ell = \{ t\vec{v} : t \in \mathbb{R} \}}$$

Let



$$\ell = \{ A + t\vec{v} : t \in \mathbb{R} \}$$

$\rightarrow \quad \vec{\ell}(t) = \vec{a} + t\vec{v}$ parametrize the line

$\vec{a}$: point on the line $\vec{v}$: vector in the direction of the line

Ex Find a parametric equation of the line ℓ in $\mathbb{R}^3$ that passes through $(3,2,-2)$ in the direction $\hat{\imath} - \hat{\jmath} + 2\hat{k}$.

$$\vec{\ell}(t) = \vec{a} + t\vec{v}$$

$$\vec{\ell}(t) = (3,2,-2) + t(1,-1,2)$$

$$= \boxed{(3+t,\ 2-t,\ -2+2t)}$$

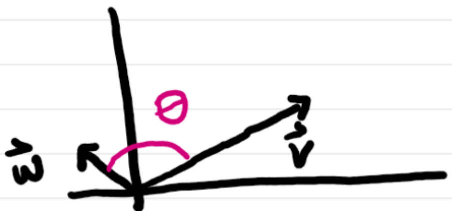
Dot Products Let $\vec{v} = (v_1, \dots, v_n)$ & $\vec{w} = (w_1,$

then $\boxed{\vec{v} \cdot \vec{w} = v_1 w_1 + \cdots + v_n w_n}$

How is this Useful?

$$\vec{v} \cdot \vec{w} = \|\vec{v}\| \cdot \|\vec{w}\| \cos(\theta)$$

where θ is the angle between $\vec{v}$ & $\vec{w}$



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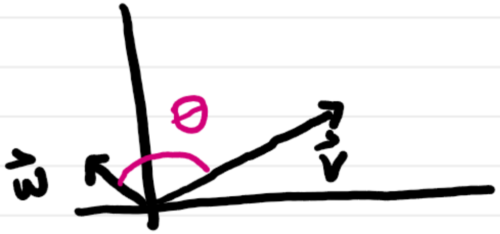
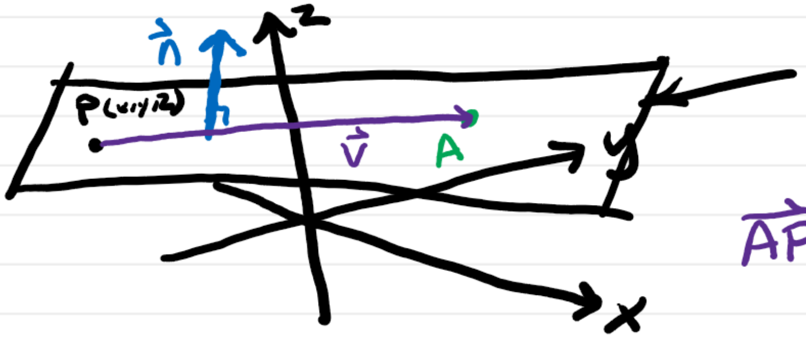


Two vectors are orthogonal (or perpendicular) if the angle between $\vec{v}$ & $\vec{w}$ is $90°$ (or $\pi/2$ radians)

$$\boxed{\vec{v}\cdot\vec{w} = \|\vec{v}\|\cdot\|\vec{w}\|\cos\left(\frac{\pi}{2}\right) = 0}$$

Equation of a Plane in Space



plane $\left\{\begin{array}{l} \bullet\ A(x_0,y_0,z_0) \text{ is a point on the plane} \\ \bullet\ \vec{n}=(a,b,c) \text{ is a vector perpendicular to plane} \end{array}\right.$

$$\vec{AP} = \vec{v} = (x-x_0,\ y-y_0,\ z-z_0)$$

$$\vec{v}\cdot\vec{n} = 0$$

$$a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$$

$$ax - ax_0 + by - by_0 + cz - cz_0 = 0$$

$$ax + by + cz = ax_0 + by_0 + cz_0$$

$$ax+by+cz = \underbrace{ax_0+by_0+cz_0}_{d}$$

$\vec{n}$

$$\boxed{ax+by+cz=d}$$ Equation of the plane

Ex Find the parametric equation for the line through (2,4,6) that is perpendicular to the plane $x-y+3z=7$.

$$\vec{\ell}(t) = \vec{a} + t\vec{v}$$

(2,4,6) (1,-1,3)

$$\vec{\ell}(t) = (2,4,6) + t(1,-1,3)$$
$$= \boxed{(2+t,\ 4-t,\ 6+3t)}$$

input outputs

#11 Consider the function $f(x) = (x^2, 1-x)$

input outputs

Its graph: $\{(x, x^2, 1-x)\} \subseteq \mathbb{R}^3$ dimension = dimension input + output

Its image: $\{(x^2, 1-x)\} \subseteq \mathbb{R}^n$ dimension output

outputs

Its level set at $\vec{a}$: $\{x : f(x) = \vec{a}\} \subseteq \mathbb{R} = \mathbb{R}^1$ dimension input

inputs

dimension = dimension input + output

Its graph: $\{(x, x^2, 1-x)\} \subseteq \mathbb{R}^3$

Its image: $\{(x^2, 1-x)\} \subseteq \mathbb{R}^n$ dimension output

outputs

Its level set at $\vec{a}$: $\{x : f(x) = \vec{a}\} \subseteq \mathbb{R} = \mathbb{R}^1$ dimension input

inputs

$$f(x,y) = (x^2, x-y, xy)$$

Graph: $\{(x, y, x^2, x-y, xy)\} \subseteq \mathbb{R}^5$

Image: $\{(x^2, x-y, xy)\} \subseteq \mathbb{R}^3$

Level set at $\vec{a}$: $\{(x,y) : f(x,y) = \vec{a}\} \subseteq \mathbb{R}^2$

Limits & Continuity for Multivariable Functions

Let $f : \mathcal{U} \subseteq \mathbb{R}^m \to \mathbb{R}$ be a real-valued function of m variables. We say

$$\lim_{\vec{x} \to \vec{a}} f(\vec{x}) = L$$

If for all $\epsilon > 0$ there exists a $\delta > 0$ such that $\|\vec{x} - \vec{a}\| < \delta$ implies $|f(\vec{x}) - L| < \epsilon$

"No matter which direction $\vec{x}$ approaches $\vec{a}$, $f(\vec{x})$ gets arbitrarily close to L"

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Ex Show that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{y^2}{\sqrt{x^2+y^2}} = 0$

Let $\epsilon > 0$. Let $\delta = \epsilon$. If $\|\vec{x} - 0\| < \delta \leftarrow$

$\|\vec{x} - 0\| = \|\vec{x}\| = \sqrt{x^2+y^2} < \delta$

$$|f(\vec{x}) - 0| = |f(\vec{x})| = \left|\frac{y^2}{\sqrt{x^2+y^2}}\right| = \frac{y^2}{\sqrt{x^2+y^2}} \leq \frac{x^2+y^2}{\sqrt{x^2+y^2}} = \sqrt{x^2+y^2} < \delta = \epsilon$$

Thus, $\boxed{\displaystyle\lim_{(x,y)\to(0,0)} \frac{y^2}{\sqrt{x^2+y^2}} = 0}$ $\square$

A function $f: \mathcal{U} \subseteq \mathbb{R}^m \to \mathbb{R}$ is continuous at $\vec{x} = \vec{a}$ if

A function $f: \mathcal{U} \subseteq \mathbb{R}^m \to \mathbb{R}$ is <u>continuous</u> at $\vec{x} = \vec{a}$ if & only if

① $\lim_{\vec{x} \to \vec{a}} f(\vec{x})$ exists.

② $f(\vec{a})$ exists.

③ $\lim_{\vec{x} \to \vec{a}} f(\vec{x}) = f(\vec{a})$.

<u>Ex</u> Find all points of discontinuity of

$$f(x,y) = \begin{cases} \dfrac{x^2 y}{x^4 + y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

The function $\dfrac{x^2 y}{x^4+y^2}$ is continuous when $x^4 + y^2 \neq 0$

$x^4 + y^2 = 0 \Rightarrow \underbrace{y^2}_{\text{positive}} = -\underbrace{x^4}_{\text{positive}}$ only when $(x,y) = (0,0)$

Just need to check continuity at $(0,0)$:

Just need to check continuity at (0,0):

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2} \overset{?}{=} 0$$

→ Choose to approach (0,0) along x=0 (y-axis)

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2} = \lim_{y\to 0} \frac{0}{y^2} = \lim_{y\to 0} 0 = 0 \checkmark$$

→ Choose to approach (0,0) along y=x

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2} = \lim_{x\to 0} \frac{x^3 \cdot 1/x^2}{(x^4+x^2)\cdot 1/x^2} = \lim_{x\to 0} \frac{x}{x^2+1} = 0 \checkmark$$

→ Choose to approach (0,0) along $y=x^2$

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2} = \lim_{x\to 0} \frac{x^4}{x^4+x^4} = \lim_{x\to 0} \frac{x^4}{2x^4} = \lim_{x\to 0} \frac{1}{2} = \frac{1}{2}$$

$\boxed{\text{So } f \text{ is not continuous at } (0,0)}$

The cross product of $\vec{v} = \langle v_1, v_2, v_3\rangle$ & $\vec{w} = \langle w_1, w_2, w_3\rangle$ is a vector $\vec{v}\times\vec{w}$

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$$\vec{v} \times \vec{w} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = (v_2 w_3 - v_3 w_2)\hat{i} - (v_1 w_3 - v_3 w_1)\hat{j} + (v_1 w_2 - v_2 w_1)\hat{k}$$

Area of a parallelogram spanned by $\vec{v}$ & $\vec{w}$
$\|\vec{v} \times \vec{w}\|$