Week 4 7PM

Tuesday, April 21, 2020 6:59 PM

Properties of Derivatives

Suppose $F: \mathbb{R}^m \to \mathbb{R}^n$ & $G: \mathbb{R}^m \to \mathbb{R}^n$ are differentiable at $\vec{a} \in \mathbb{R}^m$. Then

(1) $D(F \pm G)(\vec{a}) = DF(\vec{a}) \pm DG(\vec{a})$

(2) For $c \in \mathbb{R}$, $D(cF)(\vec{a}) = c\,DF(\vec{a})$

(3) $D(FG)(\vec{a}) = G(\vec{a})DF(\vec{a}) + F(\vec{a})DG(\vec{a})$

(4) If $G(\vec{a}) \neq 0$ then

$$D\left(\frac{F}{G}\right)(\vec{a}) = \frac{G(\vec{a})\,DF(\vec{a}) - F(\vec{a})\,DG(\vec{a})}{G^2(\vec{a})}$$

(5) $\boxed{\text{Chain Rule}}$ Suppose $F: \mathbb{R}^m \to \mathbb{R}^n$ is differentiable at $\vec{a}$ & $G: \mathbb{R}^n \to \mathbb{R}^p$ is differentiable at $F(\vec{a})$

Then

$$\longrightarrow D(G \circ F)(\vec{a}) = DG(F(\vec{a})) \cdot DF(\vec{a})$$

matrix multiplication

Ex Let $F: \mathbb{R}^2 \to \mathbb{R}^3$ be given by $\vec{F}(x,y) = (\overset{F_1}{x^3+y}, \overset{F_2}{e^{xy}}, \overset{F_3}{2+xy})$

& let $G: \mathbb{R}^3 \to \mathbb{R}^2$ be given by $G(u,v,w) = (\underbrace{u^2+v}_{G_1}, \underbrace{uv+w^3}_{G_2})$

Compute $D(G \circ F)(0,1)$

By the chain rule

$$D(G \circ F)(0,1) = DG(F(0,1)) \cdot DF(0,1)$$

$$F(0,1) = (0^3+1, e^{0\cdot 1}, 2+0\cdot 1) = (1,1,2)$$

$$DG = \begin{bmatrix} \partial G_1/\partial u & \partial G_1/\partial v & \partial G_1/\partial w \\ \partial G_2/\partial u & \partial G_2/\partial v & \partial G_2/\partial w \end{bmatrix} = \begin{bmatrix} 2u & 1 & 0 \\ v & u & 3w^2 \end{bmatrix}$$

$$DG(\underset{u}{1}, \underset{v}{1}, \underset{w}{2}) = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 12 \end{bmatrix}$$

u " v " w

$\begin{bmatrix} \; & \; & \; \\ 1 & 1 & 12 \end{bmatrix}$

$$DF = \begin{bmatrix} \partial F_1/\partial x & \partial F_1/\partial y \\ \partial F_2/\partial x & \partial F_2/\partial y \\ \partial F_3/\partial x & \partial F_3/\partial y \end{bmatrix} = \begin{bmatrix} 3x^2 & 1 \\ ye^{xy} & xe^{xy} \\ y & x \end{bmatrix}$$

$$DF(0,1) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 0 \end{bmatrix}$$

$$\rightarrow D(G \circ F)(0,1) = DG(F(0,1)) \cdot DF(0,1)$$

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 12 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 0 \end{bmatrix}$$

$$= \boxed{\begin{bmatrix} 1 & 2 \\ 13 & 1 \end{bmatrix}}$$

Another way to think of the Chain Rule

Let $f: \mathbb{R}^3 \to \mathbb{R}$ & $c: \mathbb{R} \to \mathbb{R}^3$ where $c(t) = (x(t), y(t), z(t))$ Then

$$D(f \circ c)(\vec{a}) = Df(c(a))\, Dc(a)$$

$$= \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{bmatrix}_{c(a)} \begin{bmatrix} dx/dt \\ dy/dt \\ dz/dt \end{bmatrix} = \boxed{\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\cdot\frac{dz}{dt}}$$

<u>Ex</u> (Similar to HW 3 #2)

Let $F(u,v)$ be a function of 2 variables

Let $F_u(u,v) = G(u,v)$ & $F_v(u,v) = H(u,v)$. Find

$f'(x)$ for $f(x) = F(5x, x^3)$

Let $F_u(u,v) = G(u,v)$ ...

$f'(x)$ for $f(x) = F(5x, x^3)$

$F(u,v)$ $\quad c(x) = (u(x), v(x))$

$u(x) = 5x \quad v(x) = x^3$

$$f'(x) = D(F \circ c) = \underbrace{\frac{\partial F}{\partial u}}_{F_u(u,v) = G(u,v)} \cdot \frac{du}{dx} + \underbrace{\frac{\partial F}{\partial v}}_{F_v(u,v) = H(u,v)} \cdot \frac{dv}{dx}$$

$$\frac{du}{dx} = 5 \qquad \frac{dv}{dx} = 3x^2$$

$$\boxed{f'(x) = G(u,v) \cdot 5 + H(u,v) \cdot 3x^2}$$

Contour Diagrams

Let $f: \mathbb{R}^m \to \mathbb{R}$

Level Set of $c = \{(x_1, \dots, x_m) \in \mathbb{R}^m : f(x_1, \dots, x_m) = c\}$

Level sets in the xy-plane form a contour diagram

Ex Describe the contour diagram of $f(x,y) = 3x - 2y + 1$ ←

Level Sets

$\boxed{c = 1}$ $\quad 1 = 3x - 2y + 1$

$2y = 3x$

$y = \frac{3}{2}x$ ← line

$\boxed{c = 2}$ $\quad 2 = 3x - 2y + 1$

$2y = 3x - 1$

→ $y = \frac{3}{2}x - \frac{1}{2}$ ← line

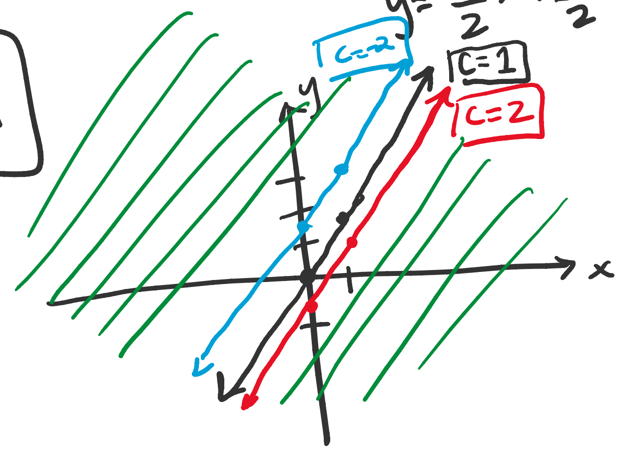
$\boxed{c = -2}$ $\quad -2 = 3x - 2y + 1$

$\boxed{C=-2}$ $\quad -2 = 3x-2y+1$

$2y = 3x+3$ line

$y = \frac{3}{2}x + \frac{3}{2}$

Contour Diagram



Directional Derivative

Let $f: \mathbb{R}^2 \to \mathbb{R}$. The directional derivative of f at $\vec{p} = (a,b)$ in the direction of the unit vector $\vec{u} = (u,v)$ is given by

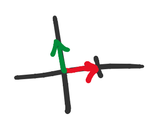
$$D_{\vec{u}} f(a,b) = \frac{d}{dt} f(\vec{p} + t\vec{u}) \Big|_{t=0}$$

$$\boxed{D_{\vec{u}} f(a,b) = f_x(a,b) \cdot u + f_y(a,b) \cdot v}$$

Partial Derivatives are directional derivatives

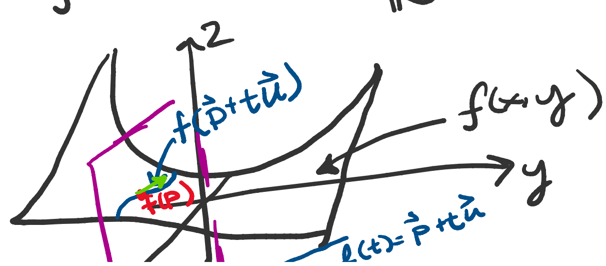
f_x is the directional derivative of f in the direction of the positive x-axis

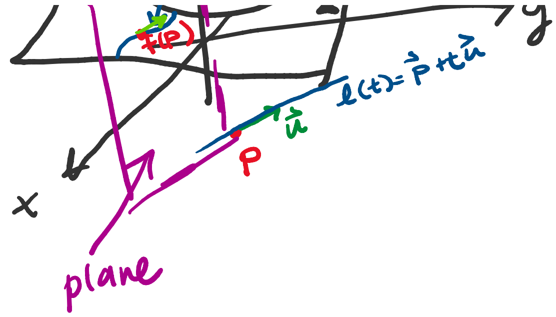
$f_x = D_{(1,0)} f$



$f_y = D_{(0,1)} f$

$\mathbb{R}^3$





<u>Ex</u> Compute the directional derivative of $f(x,y) = x^2 + 3xy$ in the direction of $3\hat{\imath} + 4\hat{\jmath}$ at $p = (2,-1)$

Not a unit vector

$D_{\vec{u}} f(2,-1)$

Normalize $\vec{v} = 3\hat{\imath} + 4\hat{\jmath}$: $\dfrac{\vec{v}}{\|\vec{v}\|} = \vec{u}$

$$\frac{\vec{v}}{\|\vec{v}\|} = \frac{3\hat{\imath} + 4\hat{\jmath}}{\sqrt{9+16}} = \frac{3\hat{\imath} + 4\hat{\jmath}}{\sqrt{25}} = \frac{3}{5}\hat{\imath} + \frac{4}{5}\hat{\jmath} = \vec{u}$$

$$D_{\vec{u}} f(2,-1) = f_x(2,-1) \cdot u + f_y(2,-1) \cdot v$$

$f_x = 2x + 3y$ $\qquad$ $f_y = 3x$

$f_x(2,-1) = 4 - 3 = 1$ $\qquad$ $f_y(2,-1) = 6$

$$D_{\vec{u}} f(2,-1) = 1 \cdot \frac{3}{5} + 6 \cdot \frac{4}{5} = \frac{3}{5} + \frac{24}{5} = \frac{27}{5}$$