Week 7 7PM

Tuesday, May 12, 2020 6:58 PM

(4) Consider the function

$$f(x,y) = \begin{cases} \boxed{\dfrac{xy^3}{x^4+y^4}} & \text{if } (x,y) \neq (0,0) \\ \boxed{0} & \text{if } (x,y) = (0,0) \end{cases}$$

(i) Show that $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist.

Along the path $x=0$:

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{y\to 0} f(0,y)$$
$$= \lim_{y\to 0} \frac{0\cdot y^3}{0^4+y^4}$$
$$= \lim_{y\to 0} \frac{0}{y^4}$$
$$= \lim_{y\to 0} 0$$
$$= \boxed{0}$$

Along the path $y=x$:

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 0} f(x,x)$$
$$= \lim_{x\to 0} \frac{x\cdot x^3}{x^4+x^4}$$
$$= \lim_{x\to 0} \frac{\cancel{x^4}}{2\cancel{x^4}}$$
$$= \lim_{x\to 0} \frac{1}{2}$$
$$= \boxed{\frac{1}{2}}$$

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(ii) Show that $\frac{\partial f}{\partial x}(0,0) = 0$

$$\frac{\partial f}{\partial x}(0,0) = \left. \frac{y^3(x^4+y^4) - xy^3 \cdot 4x^3}{(x^4+y^4)^2} \right|_{(0,0)} = \frac{0}{0} \neq 0$$

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Instead,

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h \to 0} \frac{f((0,0) + h(1,0)) - f(0,0)}{h}$$

$$= \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h}$$

$$= \lim_{h \to 0} \frac{f(h,0)}{\boxed{h}}$$

$$= \lim_{h \to 0} \frac{1}{\boxed{h}} \cdot \frac{h \cdot 0^3}{h^4 + 0^4}$$

$$= \lim_{h \to 0} \frac{1}{h} \cdot \frac{0}{h^4}$$

$$= \lim_{h \to 0} \frac{1}{h} \cdot 0$$

$$= \lim_{h \to 0} 0$$

$$= \boxed{0}$$

5. Let $f(x,y) = ye^{x^2+y}$

(i) Find an equation of the tangent plane of the graph of f at $(1, 1, e^2)$.

$$z = z_0 + \left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} (x - x_0) + \left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} (y - y_0)$$

$(x_0, y_0, z_0) = (1, 1, e^2)$

$(x_0, y_0, z_0) = (1, 1, e^2)$

$\rightarrow \frac{\partial f}{\partial x} = 2xy\,e^{x^2+y}$ $\qquad \frac{\partial f}{\partial y} = ye^{x^2+y} + e^{x^2+y}$

$\rightarrow \left.\frac{\partial f}{\partial x}\right|_{(1,1)} = 2\cdot1\cdot1\cdot e^{1^2+1} = 2e^2$ $\qquad \left.\frac{\partial f}{\partial y}\right|_{(1,1)} = 1\cdot e^{1^2+1} + e^{1^2+1} = e^2+e^2 = 2e^2$

$$\boxed{z = e^2 + 2e^2(x-1) + 2e^2(y-1)} \leftarrow$$

(ii) Find a unit vector $\vec{u}$ so that at the point (1,1) f decreases most rapidly along $\vec{u}$.

$$\vec{u} = \frac{-\nabla f(1,1)}{\|\nabla f(1,1)\|}$$

$\nabla f = (\partial f/\partial x, \partial f/\partial y) = (2xye^{x^2+y}, ye^{x^2+y} + e^{x^2+y})$

$\nabla f(1,1) = (2e^2, 2e^2)$

$\|\nabla f(1,1)\| = \sqrt{(2e^2)^2 + (2e^2)^2} = \sqrt{4e^4+4e^4} = \sqrt{8e^4} = 2e^2\sqrt{2}$

$$\vec{u} = -\frac{(\cancel{2e^2}, \cancel{2e^2})}{\cancel{2e^2}\sqrt{2}} = \boxed{\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)}$$

(6) Use the method of Lagrange multiplier to find the maximum & minimum of the function
$$f(x,y,z) = y^2 + xz$$
with the constraint $x^2+y^2+z^2=4$

$\nabla f = (\partial f/\partial x, \partial f/\partial y, \partial f/\partial z) = (z, 2y, x)$

$\rightarrow g(x,y,z) = x^2+y^2+z^2 = 4$

$\nabla g = (2x, 2y, 2z)$

$\nabla g = (2x, 2y, 2z)$

If $\nabla g = \vec{0}$ then $(2x, 2y, 2z) = (0,0,0)$

$\Rightarrow (x,y,z) = (0,0,0)$

$g(0,0,0) = 0^2 + 0^2 + 0^2 = 0 \neq 4$

So when $g(x,y,z) = 4$ we don't have $\nabla g = \vec{0}$

Solve $\nabla f = \lambda \nabla g$

$z = \lambda 2x \qquad \boxed{2y = \lambda 2y} \qquad x = \lambda 2z$

$\boxed{\text{If } y \neq 0:}$ $\dfrac{2y}{2y} = \dfrac{\lambda 2y}{2y}$ $\boxed{1 = \lambda}$

$\boxed{z = 2x} \qquad x = 2z$

$x = 2(2x) = 4x$

$x = 4x \rightarrow \boxed{x = 0}$

$z = 2x = 2 \cdot 0 = 0 \quad \boxed{z = 0}$

$x^2 + y^2 + z^2 = 4$

$0^2 + y^2 + 0^2 = 4$

$y^2 = 4$

$y = \pm 2$

Candidates: $(0,2,0)$ & $(0,-2,0)$

$f(x,y,z) = y^2 + xz$

$\left.\begin{array}{l} f(0,2,0) = 2^2 + 0 = 4 \\ f(0,-2,0) = (-2)^2 + 0 = 4 \end{array}\right\}$ Max

$\boxed{\text{If } y = 0}$ $\boxed{x = \lambda 2z}$ & $\boxed{z = \lambda 2x}$

$\dfrac{xz}{xz} = \dfrac{\lambda^2 \cdot 4 \cdot xz}{xz}$

$$\frac{xz}{xz} = \frac{\lambda^2 \cdot 4 \cdot xz}{xz}$$

$$1 = \lambda^2 \cdot 4$$

$$\frac{1}{4} = \lambda^2$$

$$\lambda = \pm \frac{1}{2}$$

$\boxed{\lambda = 1/2}$ $\quad x = \lambda 2z \quad x = \frac{1}{2} \cdot 2 \cdot z$

$$x = z$$

$$x^2 + y^2 + z^2 = 4$$

$$x^2 + 0^2 + z^2 = 4$$

$$x^2 + z^2 = 4$$

$$x^2 + x^2 = 4$$

$$2x^2 = 4$$

$$x^2 = 2$$

$$x = \pm\sqrt{2}$$

Candidates: $(\sqrt{2}, 0, \sqrt{2})$ & $(-\sqrt{2}, 0, -\sqrt{2})$

$$f(x,y,z) = y^2 + xz$$

$$\left.\begin{aligned} f(\sqrt{2}, 0, \sqrt{2}) &= 0^2 + (\sqrt{2})(\sqrt{2}) = 2 \\ f(-\sqrt{2}, 0, -\sqrt{2}) &= 0^2 + (-\sqrt{2})(-\sqrt{2}) = 2 \end{aligned}\right\}$$

$\boxed{\lambda = -1/2}$ $\quad x = \lambda \cdot 2z = -\frac{1}{2} \cdot 2z \quad x = -z$

$$x^2 + y^2 + z^2 = 4$$

$$x^2 + z^2 = 4$$

$$x^2 + (-x)^2 = 4$$

$$x^2 + x^2 = 4$$

$$2x^2 = 4$$

$$x^2 = 2$$

$$x = \pm\sqrt{2}$$

Candidates: $(\sqrt{2}, 0, -\sqrt{2})$ & $(-\sqrt{2}, 0, \sqrt{2})$

$$f(x,y,z) = y^2 + xz$$

$$\left.\begin{aligned} f(\sqrt{2}, 0, -\sqrt{2}) &= 0^2 + (\sqrt{2})(-\sqrt{2}) = -2 \\ f(-\sqrt{2}, 0, \sqrt{2}) &= 0^2 + (-\sqrt{2})(\sqrt{2}) = -2 \end{aligned}\right\} \text{Min}$$

$f(-\sqrt{2}, 0, \sqrt{2}) = 0^2 + (-\sqrt{2})(\sqrt{2}) = -2$

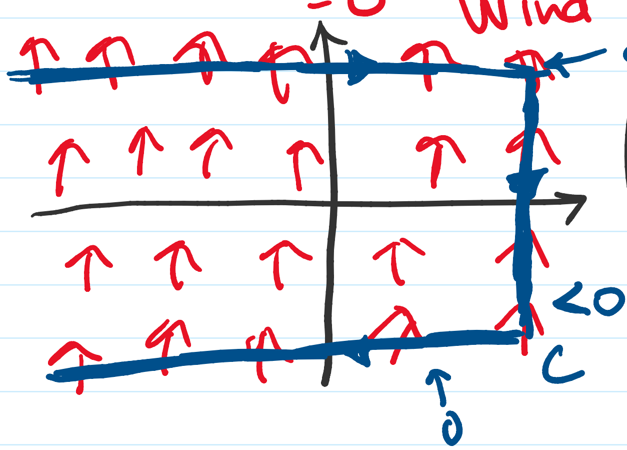
$$\boxed{\begin{array}{ll} \underline{\text{Max}} & f(0, \pm 2, 0) = 4 \\ \underline{\text{Min}} & f(\pm\sqrt{2}, 0, \mp\sqrt{2}) = -2 \end{array}}$$

HW6 Suppose you are flying in a plane. We want to know if the flow of the wind is going with the path of the plane or against it.

Let $\vec{F}$ be the vector field representing the wind currents & let $\vec{p}(t)$ be the path of your plane. Then

$$\underbrace{\int_{\text{takeoff}}^{\text{landing}} \underbrace{\vec{F}(\vec{p}(t))}_{\text{wind current}} \cdot \underbrace{\vec{p}\,'(t)}_{\text{velocity of plane}} \, dt}_{\text{net accumulation of wind during your flight}}$$

>0 wind was in the direction of the path
<0 flying against the wind
$=0$ Wind was perpendicular



$$\boxed{\int_C \vec{F} \cdot d\vec{p} < 0}$$