Diagrammatic Presentations of Affine Type A Subfactor Planar Algebras of Index 4

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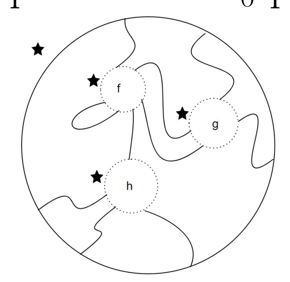
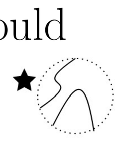
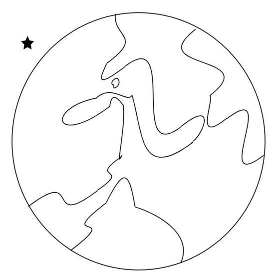
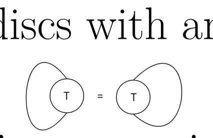
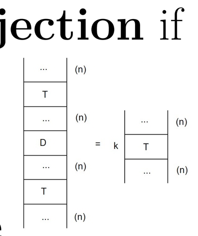
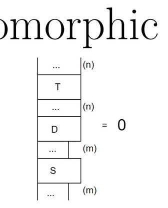
Introduction

The category of representations of a finite group is an example of a fusion category. Within this type of category we can take linear combinations of morphisms, and tensor products and direct sums of objects. A planar algebra can be constructed from a unitary fusion category with the advantage that the structure is now conveniently encoded by diagrams in the plane. Conversely, given a *subfactor* planar algebra, we can construct a fusion category of projections. The Kuperberg program is a proposal to give a presentation by generators and relations for every subfactor planar algebra. In this poster I contribute to this program by giving presentations for the $\tilde{A}_{2n-1}$ subfactor planar algebras of index 4.

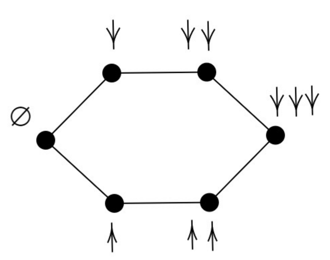
Definitions

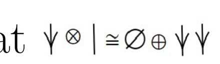
- A **planar algebra** $\mathcal{P}$ associates a vector space $\mathcal{P}_k$ and a linear map $\mathcal{P}(T) : \mathcal{P}_{k_1} \otimes ... \otimes \mathcal{P}_{k_r} \to \mathcal{P}_{k_0}$ to each planar tangle diagram T with internal discs with $k_1, ..., k_r$ points and k_0 points on the external disk.

  For example:  gives a map from $\mathcal{P}_6 \otimes \mathcal{P}_4 \otimes \mathcal{P}_4 \to \mathcal{P}_6$. We could have an element of $\mathcal{P}_6$ be  and plug it into f, take elements  and  in $\mathcal{P}_4$ and plug them into g and h respectively, and get , which is an element of $\mathcal{P}_6$
- A **subfactor planar algebra** is a planar algebra that must satisfy the following additional rules:
  - $\mathcal{P}_0$ must be one-dimensional.
  - Only spaces for discs with an even number of boundary points are nonzero.
  - For each $T \in \mathcal{P}_2$, . We call this property being **spherical**.
  - There must be an anti-linear adjoint operation $* : \mathcal{P}_k \to \mathcal{P}_k$ such that the sesquilinear form given by $< x, y >= \mathrm{tr}(y^* x)$ is positive-definite.
- A closed circle in a subfactor planar algebra is in $\mathcal{P}_0$, so it is a complex number. In particular, it is a multiple of the empty diagram. The square of this number is called the **index**.
- An element $T \in \mathcal{P}_{2m}$ of a planar algebra is called a **minimal projection** if $T^2 = T$, $T^* = T$, and for any $D \in \mathcal{P}_{2m}$, there exists a $k \in \mathbb{C}$ where 
- A planar algebra is called **semisimple** if every projection is a direct sum of minimal projections and if for any two non-isomorphic minimal projections $S \in \mathcal{P}_{2m}$ and $T \in \mathcal{P}_{2n}$ we have for $D \in \mathcal{P}_{m+n}$, 
- The **principal graph** of a semisimple planar algebra has as vertices the isomorphism classes of minimal projections and encodes a 'multiplication table' of these elements.

The Principal Graph

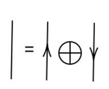
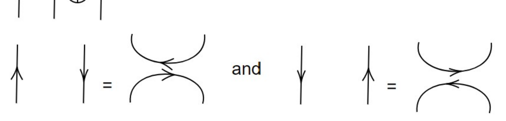
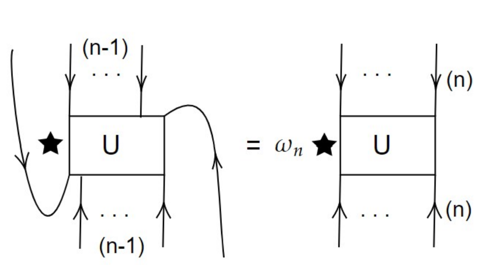
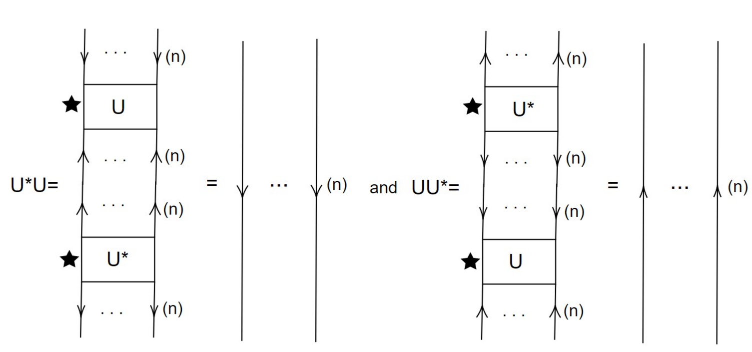
Subfactor planar algebras of index 4 with finite principal graphs must have an affine Dynkin diagrams as that graph. The principal graph works as a 'multiplication table' in the following way: every vertex's projection tensor a single strand is isomorphic to the direct sum of its neighbors. For example, for $\tilde{A}_5$ we may have a principal graph:



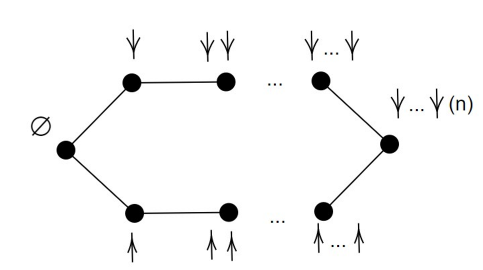
which gives that 

Presentations of $\tilde{A}_{2n-1}$

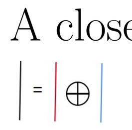
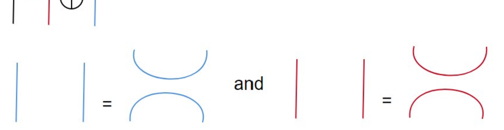
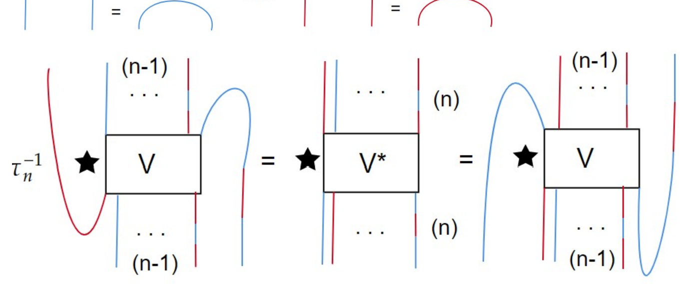
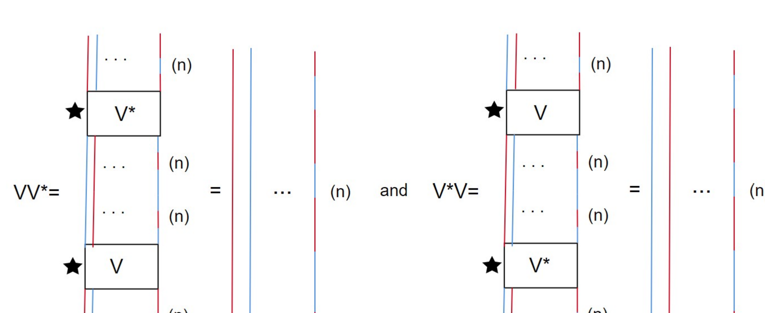
Theorem 1 (M.) Fix n. Let ω_n be a $2n$th root of unity. Let $\mathcal{PA}(U)$ be the planar algebra generated by a single box U with $2n$ strands oriented inwards, modulo the following relations:

1. A closed circle equals 2
2. 
3. 
4. 
5. 

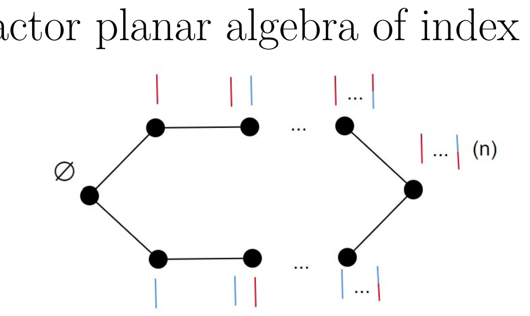
Then this is an $\tilde{A}_{2n-1}$ subfactor planar algebra of index 4 with principal graph:



Theorem 2 (M.) Fix n. Let τ_n be an nth root of unity. Let $\mathcal{PA}(V)$ be the planar algebra generated by a single box V with $2n$ strands alternating blue/red, modulo the following relations:

1. A closed circle equals 2
2. 
3. 
4. 
5. 

Then this is an $\tilde{A}_{2n-1}$ subfactor planar algebra of index 4 with principal graph:



Applications

The goal of the Kuperberg program is to understand as much as possible about each subfactor planar algebra just based upon the combinatorics of the diagrams. This way, we can easily work with the planar algebra and apply results obtained to their corresponding fusion categories. Additionally, subfactor planar algebras arise in functional analysis. A *factor* is a von Neumann algebra with a trivial center and a *subfactor* is then a unital inclusion of factors. Surprisingly, we have the following two theorems:

Theorem (Popa) All subfactor planar algebras arise from subfactors.

Theorem (Jones) The planar algebras coming from subfactors are subfactor planar algebras.

As a result, having a diagrammatic presentation of $\tilde{A}_{2n-1}$ also lets us not need to appeal to von Neumann algebras as well.

Future Work

Theorem (Popa) The number of subfactor planar algebras associated to each of the corresponding Dynkin diagrams is given by:

Principal Graph	$\tilde{A}_{2n-1}$ (2n vertices)	$\tilde{A}_\infty$	$\tilde{D}_n$ (n+1 vertices)	$\tilde{E}_6$	$\tilde{E}_7$	$\tilde{E}_8$	A_∞	$\tilde{D}_\infty$
Number of distinct subfactor planar algebras	n	1	n-2	1	1	1	1	1

Using this classification, I am interested in giving presentations for the rest of these cases. The $\tilde{A}_\infty$ case will follow from the presentations of $\tilde{A}_{2n-1}$. I have already done much work towards the $\tilde{D}_\infty$ case. Further, some of these subfactor planar algebras are contained in others, so I wish to formally prove these containments using the diagrammatic presentations. Additionally, this table shows that there are only n distinct $\tilde{A}_{2n-1}$ subfactor planar algebras, so I would like to find which of the ones I've listed in Theorem 1 and 2 are non-isomorphic. Lastly, I have proven that these are the presentations of index 4, but I also want to prove that given these presentations (without the first relation), the index *must* be 4.

References and Acknowledgements

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